Sloan School of Management

Ph.D. General Examination
Operations Research

February 15, 1967
9 - 12 a.m.

The examination is closed book. In case of doubt about question interpretation, make the most reasonable interpretation you can and note accordingly. There are a total of 100 points on the exam and the value of each question is indicated. Please do each exam in a separate book.

✓ 1. (12 points) Assume that we are given the primal problem $Ax=b$, $x \geq 0$, Max $z = cx$, and an optimal solution w_0 to the dual of the problem. We now form a new primal problem by adding ($\lambda \neq 0$) times constraint k to constraint r. What is an optimal solution to the dual of this new problem?

$z^* = 5^*$

2. (12 points) Is $x_1=2$, $x_2=0$, $x_3=1$ an optimal solution to the following problem? Give support for your answer.

$$\min \quad x_1 + 3x_2 + x_3 \quad \rightarrow 2+1=3$$

$$\text{s.t.} \quad x_1 - x_2 - 2x_3 \geq 0 \qquad 2-2=0 \geq 0$$

$$x_1 + x_2 + x_3 \geq 2 \qquad 2+1=3 \geq 2$$

$$-x_1 + 2x_2 + 3x_3 \geq 1 \qquad -2+3=1 \geq 1$$

$$x_1, x_2, x_3 \geq 0$$

$z_j - c_j < 0$ all 5 vars

$3x_2 + 4x_3 \geq 3$

$x_2 + x_3 \geq 1$

$3x_2 + 3x_3 \geq 3$

$x_3 \geq 1$

opt.

3. (12 points) Consider the following problem.

$$\min \quad z = 6x_1 + 4x_2 + x_3 + y_3x_4$$

s.t.

$$x_1 + x_2 - 4x_3 + y_1x_4 = 5$$

$$-x_1 + x_2 - x_3 + y_2x_4 = 1$$

$$y_1 + 2y_2 + 3y_3 = 2$$

$$x_i \geq 0, \; y_i \geq 0.$$

$z_j < c_j$

~~min $c_j - z_j$~~

$z_j - c_j$

min $(c_j - z_j)$

max $(z_j - c_j)$ ✓

$-y_3 + c_B B^{-1} a_4$

Discuss how this problem can be solved by use of the simplex method.

max $z_4 - c_4$ st consts on y.

$\pi_i(t)$ & x_t $\qquad g_n(x)$

$$\pi_n''(t) = \frac{P(x_t = x \mid M_n)\,\pi_n(t)}{\sum_{i=1}^{N}[\qquad]}$$

$$\pi_n'(t+1) = \sum_{i=1}^{N} \pi_i''(t)\, P_{in}$$

s.s. Π

$$C(y,\pi) = \min_{z} E\{g(\quad) + aC(y,\pi)\}$$

assume π, z, y

$y, \pi(1)$ $\qquad a^k = 0$

$f_K \equiv 0$

4. (32 points) Consider a single item inventory system. Let $\{\tilde{x}_t = x\}$ denote the event that x units are demanded in period t, t=1,2,... . Suppose that $\tilde{x}_t$ has one of N possible mass functions. Denote the different m.f.'s by M_n, n=1,2,...,N, and let

$$g_n(x) = P(\tilde{x}_t = x | M_n)$$

independent of t. Assume a Markovian relationship between the m.f. in one period and the m.f. in the next. Specifically, let P_{mn} be the probability that m.f. M_n obtains in period t+1 given that M_m obtained in period t, t=1,2,..., thus, $\sum_n P_{mn} = 1$. And finally, let $\Pi_n(t)$ denote the probability that $\tilde{x}_t$ has m.f. M_n.

a. Given $\underline{\Pi}(t) = (\Pi_1(t),\ldots,\Pi_N(t))$ and $\{\tilde{x}_t = x\}$ develop an expression for $\Pi_n(t+1)$. $\pi_m''(t) = \frac{g_m(x)\,\pi_m(t)}{\sum g_m(x)\,\pi_m(t)}$ + $\pi_m(t+1) = \sum_{i=1}^{N} \pi_i''(t) P_{im}$

b. Let $\tilde{y}_t$ denote the inventory level at the start of period t. Assume that an order to replenish inventory placed at the start of period t arrives at the start of period t+1. (Thus if z units are ordered, $\tilde{y}_{t+1} = \tilde{y}_t - \tilde{x}_t + z$.) Also consider the following costs:

c_1 = unit storage cost

c_2 = unit stockout cost

c_3 = fixed order cost

c_4 = unit order costs,

$C(y,\underline{\Pi})$ = total expected discounted cost under an optimal replenishment policy (let a denote the discount factor).

Formulate the problem of finding an optimal replenishment policy as a dynamic programming problem. Comment on the approach you would take to solve the problem.

$$\min_z E_{\pi_m''(t|x_{t-1})} E_{g_m(x)} \left\{ \xi(x,y,z) + aC\left(y+z-x;\ \pi_i''(t+1|x=x)\right) \right\}$$

$$\pi_i'(t+1) = \sum \pi_j''(t) P_{ji}$$

$$\pi_i''(t+1) = \frac{g_i(x)\,\pi_i'(t+1)}{\Sigma}$$

5. (32 points) Consider a single server model with Poisson input (rate λ) and exponential service times (rate μ), $\lambda < \mu$; derive the probability distribution of the waiting time (including service). What is the expected waiting time?

$Ax = b \rightarrow w$ unrestricted
$x \geq 0 \qquad A'w \geq c'$
max $z = cx \qquad$ min $b'w$

$$\begin{array}{c|ccc}
 & x_1 & \cdots & x_m \\
w_1 & a_{11} & \cdots & a_{1m} \\
w_2 & a_{21} & \cdots & = b_k \\
\vdots & & &
\end{array}$$

$$\begin{vmatrix} 1 & -1 & -2 \\ 1 & 1 & 1 \\ -1 & 2 & 3 \end{vmatrix} = 3 - 4 + 1 - \{2 - 3 + 2\} = 0 - 1 \neq 0$$

$z_j - c_j \leq 0$

$$B = \begin{bmatrix} 1 & -1 & -2 \\ 1 & 1 & 1 \\ -1 & 2 & 3 \end{bmatrix}$$

$a = By$

$z_j = \sum c_i y$

DOCTORAL EXAMINATION

Sloan School of Management

Operations Management

February, 1967
(Closed Book)

1. The production smoothing problem, traditionally conceived, is one of converting the seasonal sales problem or production scheduling problem into a mathematical problem for which a solution or reasonable approximation is sought.

   a) Name as many alternative ways of meeting the seasonal sales problem as possible. For example, the firm could vary inventory levels to absorb the changes in demand.

   b) For some firms short run scheduling using inventories, etc., is quite costly or not feasible. What changes can be made in the long run to reduce the costs brought about by fluctuating demand for the organization's output?

   job shop
   large constr.
   extremely perishable product (newspaper)

   adv. to chg demd
   find alt uses (reuse paper)
   differentiate mkt.

2. You have 6 jobs, each of which must be processed by two machines A and B in the order AB. Processing times are given as follows:

Job	Time on A	Time on B	
1	6	6	12
2	7	5	12
3	2	8	10
4	3	5	8
5	5	1	6
6	4	7	11
	27	27+5 = 32	

   a) Determine the sequence of jobs that will minimize the elapsed time from the start of the first job to the completion of the last job.

   b) Discuss the general problem of sequencing n jobs through m machines. At least in part, address yourself to a description of the current state of the research on this problem, citing appropriate published papers.

X charts
nomograms
PERT & CPM
ass line prec rel'n
Process diagrams
graphs
assembly directions
Plant layout
blueprints
geometric area (LP)
Decision diagrams
inv levels
etc

3. Someone once said that a picture is worth a thousand words. What pictures (diagrams, charts, etc.) are useful in the theory and practice of operations management? Justify the "worth" of each picture included in your answer.

4. In a Management Science article, Drucker defines planning as "the continuous process of making present entrepreneurial (risk taking) decisions systematically and with the best possible knowledge of their futurity, organizing systematically the efforts needed to carry out these decisions, and measuring the results of these decisions against the expectations through organized, systematic feed-back".

He goes on to say that a risk taking entrepreneurial decision has the following four elements (among others):

1. Objectives - or a criterion function
2. Assumptions structure
3. Expectations eval
4. Alternative courses of action

(where one course is the possibility of doing nothing).

Given this structure, consider the following two problems that might face an industrial planner.

(a) The planner is in charge of the construction of a large paper mill. Some of the jobs are falling behind schedule. He must decide whether or not to "crash" (allocate more resources to) some of the jobs.

(b) A Manager is in charge of the distribution system of a large consumer goods manufacturer. The system contains two production facilities and six warehouses. Sales are continually increasing and it is his problem to decide on new additions to warehouse capacity (perhaps completely new warehouses).

Answer the following planning questions in the context of the two problem areas stated above.

1. What is the objective function and what are the management decision variables in the two problems? Be specific.

2. Would you actually review the problems continuously as Drucker suggests? If not, how would you determine the optimal review period?

3. What approach would you use to justify a particular solution to the problem?

4. Briefly select and defend a planning horizon for each of these problems.

February 8, 1967

General Examination
Micro Theory -- Two Hours

Answer any THREE questions (40 minutes each).

1. In a two-factor, two-product economy with fixed stocks of L and T, both industries use L and T in the same proportion when facing the same pair of factor prices. One industry is purely competitive, with constant returns to scale; but the other is a "natural monopoly," with moderately increasing returns to scale. Every household always divides its income equally between the two goods.

   (a) What kind of transformation curve will the economy have?

   (b) If the monopoly is regulated so that price equals average cost, what will this imply as to the economy's general equilibrium? Comment on the relationships among the prices and quantities of the two goods, the real wages and rents in terms of both goods, and the labor and land quantities allocated to the two industries.

   (c) Is there then any way in which the allocation of resources can then be improved? Explain.

2. An individual has an income from property of Y_0 per year. If he faces a given wage rate, at which he chooses to work t_1 hours for total wages of Y_1, what will be the comparative revenues of the following alternative taxes, when each would have the same effect on the individual's own welfare:

   (a) a lump-sum tax
   (b) a proportional tax on his wage income
   (c) a progressive tax on his wage income?

3. A monopolist faces a linear demand for his product, which is produced with just L and T subject to fixed coefficients. What can you say about his demand for L

   (a) in a short run when T is fixed in quantity, and

   (b) in the long run when T is available at a fixed price? Explain fully.

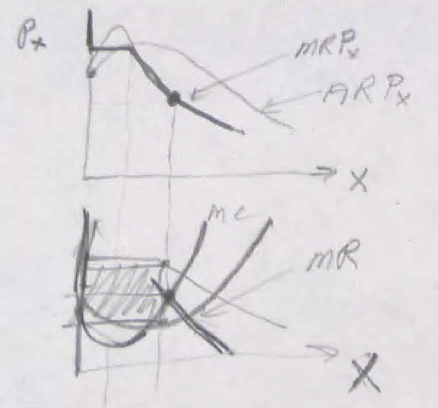
General Examination
Micro Theory

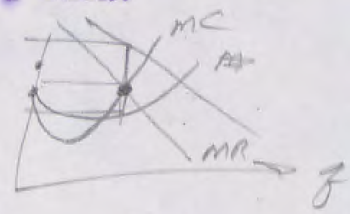
February 8, 1967

4. Discuss the welfare economics doctrines associated with at least three of the following economists:

- a. Bergson — SWF & Scitovsky contours
- b. Arrow — options for SWF & impossi
  c. Pigou — cons. surpl?
- d. Hicks-Kaldor-Scitovsky bribes

5. Discuss the similarities and differences between the problems of duopoly and bilateral monopoly. You may limit your discussion to the simpler standard instances of each.





$\cancel{\pi_1 = P_x x(y) - P_y y}$

$\pi_1 = P_x(x)x$

Ph.D. General Examination

Research Management
February 13, 1967

Answer 3 questions only:

1. What facts, considerations, and procedures should determine (or at least be taken into account) the relative level of support of basic science, applied research, development and testing in one of the following:

a. U.S. total national effort
b. A specified federal department or agency
c. A large multi-product firm

2. Outline the relevant staffing policies for a newly established government laboratory for transportation research (not including operation of any systems). Defend your recommendations.

recruitment of ... trng & obsol mix basic v applied mgt v prof. salary

3. Siwash Institute of Technology, in cooperation with the local Chamber of Commerce, wants to stimulate the growth of an industrial technical park (like Route 128). Present evidence for your advice to the planning board.

4. The XYZ Corp. R&D laboratory (the only lab in a medium size drug and chemical firm) has always been organized in functional departments. Under what conditions and for what reasons would you propose the introduction of project organization? What kind of organization and policy procedures would you advise?

1. Long, med, short run tech capabil
2. reserve capabil for econ & def strength
3. reln to privat programs (S & D oblig for social opt?)
4. opt reln basic to applied to devel
5. Educ byproducts & contin bal of manpower & res effort
6. Costs, benefits, avail opportunities, ~~[illegible]~~
7. politics
8. fed govt v. privat econ v. noble ideals as end use
9. Intl prestige
10.

General Examination
in Policy Analysis

for

C. T. Whitehead

May 1967

1. You should allow one hour for this question. Discuss briefly each of the following four propositions:

    1) It is not necessary to apply an interest rate in estimating and comparing the costs of alternative government programs.

    2) There is no satisfactory way of dealing with uncertainty about a prospective opponent's future military capabilities.

    3) Breakeven analysis is a useful technique to use when comparing alternative systems or programs.

    4) There are examples of cost-effectiveness studies where, in effect, systems design has been more important than systems analysis.

2. After developing theories of organizational goals, organizational expectations, and organizational choice, Cyert and March begin their next section as follows:

> In the course of developing the three subtheories, we have developed a relatively small number of relational concepts. In many respects, they represent the heart of our theory of business decision making. The four major concepts used in the theory are: (1) quasi resolution of conflict, (2) uncertainty avoidance, (3) problemistic search, and (4) organizational learning.

Discuss each of these four concepts. Emphasize the implications of systems on the concepts and the implications of the concepts for systems analysis.

3. Much of the current literature on the application of systems analysis discusses military planning problems. An example of this would be Quade's chapter on "Methods and Procedures" in Analysis for Military Decisions. It is clear, however, that the approach that Quade outlines would be applicable to many industrial planning problems. Consider the following problem:

   A manufacturer faced with a growing market believes it might be necessary to expand his production and distribution system. He must decide on the amount of capacity to add to his production and warehouse facilities and he must decide on the location of such additions.

   Describe the application of systems analysis to this problem. Be sure to include a discussion of the following points:

   - what is the criterion function?

   - what are the sources of uncertainty?

   - what models are applicable?

   - to what extent are existing operations research techniques useful?

Sloan School of Management

Ph.D. General Examination May, 1967

Operations Research

The examination is <u>closed book</u>. In case of doubt about question interpretation, make the most reasonable interpretation you can and note accordingly. There are a total of 100 points on the exam and the value of each question is indicated. Please do each question in a separate book.

(1) 30 points

Suppose an event E occurs at time t (t=1,2,3...) with known probability p(t). Let the "occurrence time" mass function be given by,

$$p(t) = a(1-a)^{t-1} \quad , \quad t=1,2,\ldots$$

Furthermore, let

$$Q(\tau) = P(\tilde{t} \leq \tau). \quad , \quad -\infty < \tau < \infty \quad = 1-(1-a)^{\tau} \qquad \tau > 0$$

(a) Derive an expression for $Q(\tau+1)$ in terms of $Q(\tau)$.

Now further suppose that an observation is made at time $\tau(\tau=1,2,\ldots)$, the result being a random variable $\tilde{x}(t)$ which has density function $q(x)$ if $\tilde{t} > \tau$ and $h(x)$ if $\tilde{t} \leq \tau$.

(b) Suppose at time τ, $\tilde{x}(\tau) = x$. Derive an expression for $Q(\tau+1|\tilde{x}(\tau) = x)$ in terms of $Q(\tau)$.

(2) 35 points

A random variable Y assumes one of the values, $1,2,\ldots,m$. There are two possible probability distributions of Y, viz., $\{a_k\}$ and $\{b_k\}$ where $a_k > 0$, $b_k > 0$ and $\sum_{k=1}^{m} a_k = 1$, $\sum_{k=1}^{m} b_k = 1$, Assume that the random variable Y is defined so that

$$\frac{a_1}{b_1} < \frac{a_2}{b_2} < \ldots < \frac{a_m}{b_m} \quad .$$

Suppose now that on the basis of one observed value of Y it is desired to test the hypothesis: H_a: $\{a_k\}$ is the distribution of Y versus the alternative hypotheses: H_b: $\{b_k\}$ is the distribution of Y.

A decision rule for this purpose is a set of numbers $x_1,\ldots,x_m$, $0 \leq x_j \leq 1$, such that if Y is observed equal to k, the H_a is accepted with probability x_k and H_b is accepted with probability $1-x_k$. The decision rule must be chosen so that the probability of accepting H_a when it is false (so H_b is true) does not exceed a fixed number β, $0 < \beta < 1$. Among all such rules a rule which maximizes the probability of accepting H_a when it is true will be termed optimal.

(a) Formulate the problem of finding an optimal decision rule as one in linear programming. Also formulate its dual.

(b) Show by using the principle of complementary slackness that and optimal rule $\{x_k^*\}$, say, must have the property that $x_j^* = 1$ for $j>k$ whenever $x_k^* > 0$. (This implies that for some integer K, $x_j^* = 0$ for $j<K$ and $x_j^* = 1$ for $j>K$.)

(3) 35 points

Given a single server queueing system with constant input (one every $1/\lambda$ units of time) and exponential service time (mean $1/\mu$), derive the equilibrium probabilities of an arrival finding $k (k = 0,1,\ldots)$ individuals in the system. Assume

$$\rho = \frac{\lambda}{\mu} < 1 \quad .$$

Pringle

September 16, 1966
Three hours

Sloan School of Management
Ph.D. Qualifying Exam in Operations Management

1. Discuss the problem of establishing optimal control limits on $\overline{X}$ control charts. Do not limit yourself to existing methods.

2. Quite often the problem of determining the quantity of each item to be produced in the next period is too complex to be solved in one step. It may be possible, however, to break the problem up into an aggregate decision concerning overall production, inventories and employment and a series of item decisions which allocate aggregate production to individual items.

   a. Discuss the various techniques by which the aggregate decision can be solved.

   b. Discuss techniques by which the item decisions can be made and yet be consistent with the aggregate decision rule.

3. The use of the digital computer to perform simulated experimentation on a numerical model of a complex system is an increasingly important technique in many disciplines today. It has made possible the systematic study of problems when analytic methods are not available. As with any experimental technique there are methodological difficulties connected with its efficient utilization. Four basic problems that arise are specification of functional forms, estimation of parameters, validation of the model and finally, problems of comparing output from the model when testing alternative management policies. This last area includes problems of start-up as well as the specification of equilibrium conditions.

   Discuss these problems and solutions to them in an application of simulation to one of the following problems. (References to relevant literature would be helpful.)

   1. Job Shop Scheduling

   2. The effect of investment in additional retail outlets on the cash flow of an oil company. (Assume sales are not independent from company to company.)

   3. Price and sales goal determination in a department store.

4. Recently the Domestic Fuel Oil Division of Universal Coal and Oil Company has expanded from a single tank firm or depot operation to a large multi-depot division as a result of the acquisition of a number of independent, single-depot firms. Operationally, Universal purchases the fuel oil by shipload from one of the major refineries and stores it in tanks at the depots; from the tanks it is delivered by a Universal truck to the customer, such as the homeowners. (For present purposes assume that only a single grade of fuel oil is handled by the Division.)

To date no steps have been taken to integrate the operations of the various acquisitions. For instance, each depot continues to maintain the same fleet of delivery trucks it had at the time of acquisition, and to supply the same customers it had (although the geographical areas served by different depots overlap in many instances). Typically, in the daily operation of a depot, orders are grouped by the dispatcher into individual trips and assigned to trucks on the basis of the quantities ordered, the geographical locations of the deliveries and the capacities of the trucks. In some instances a given order must be filled on the particular day in question; in others there is a period of several days in which delivery is acceptable to the customer.

While the characteristics vary among depots, the number of trucks available for delivering the grade in question lies between 1 and 10; the number of orders per day varies up to about 200; and the maximum number of orders that can be carried on a single truck in one trip varies up to about 10.

Discuss possibilities for improving the operations of the Domestic Fuel Oil Division which you would investigate. Describe any models you would include in the investigation.

2-7-66
Bergstrom
Johnson
Schuenke

Operations Management
Ph.D. Exam

1. A reoccurring problem for the manager of a production facility is to schedule production in such a way that a known requirement for a fixed number of non-defective items is satisfied by a certain date with production equipment that is not perfect. This problem goes under the general heading of "the reject allowance problem".

   a. Making all assumptions you think are appropriate give a precise formulation of the most "realistic" version of the reject allowance model that you could attack with exact mathematical methods.

   b. Now formulate the simplest version of the model formulated in (a.) that has any hope of giving you insight into the solution of (a.).

   c. Criticize the model formulated in (a.) from the operating production manager's point of view.

2. Under certain conditions, a cost minimizing formula for a lot size is:

$$L = \sqrt{\frac{2QB}{VC}} \quad \text{or, equivalently:} \quad Q = \sqrt{\frac{2RS}{I}}$$

   a. Under what conditions is this formula appropriate? Under what conditions is it inappropriate?

   b. Define B (i.e., S). Which of its components may be important? How can each of these components be estimated?

3. How would you incorporate estimates of risk into a decision model constructed to help managers make a choice among alternative capital investments? Defend your answer.

C 4. Discuss briefly the various approaches to aggregate production scheduling developed in the literature. Choose one and discuss in some detail the problems of making it operational in a particular firm. What would be your approach to dealing with these problems?

5. The Operations Manager of a large manufacturing plant (75,000 items, 6000 employees) is pleased by the returns he has earned by the application of computer technology to several of his operations. Computer explosions of manual product forecasts and computer determined shop loadings consistent with these forecasts have given him substantially better information on his operations than he had through the manual system which was superceded. He also feels he has earned substantial returns from applications in the area of payroll, accounting, inventory control, etc.

   Now he sees more applications for the computer than he can reasonably expect to accomplish in the near term. He has a Computer Systems Department made up of about 10 professionals and about 25 technicians. He would like to expand this operation but he has been unsuccessful in his attempts to hire people within the constraints of the Corporation wage structure.

   He has asked for your recommendations on the following questions:

   a. By what process should he expand his professional and technical staffs in the Computer Systems Department?

   b. How should he organize that department to insure that their efforts are being correctly allocated between the maintenance of existing programs and the development of new ones?

   c. How can he wisely choose those new applications he will have carried out from among the large set of alternatives which have been and are being defined for him?

Schwenke

Ph.D. General Examination
Production
September 15, 1965

1. (a) List the major contributions to the field of production management which have been made over the past fifteen years. Identify the contributors where possible.

   (b) Choose that contribution which you feel will be of the greatest significance.

   (c) Defend your choice.

2. Outline the role of digital simulation in production research. Cite examples from the literature and discuss the problems inherent in this type of research.

3. The traffic manager of a multi-plant, multi-warehouse system has analysed his operations for the past year by means of linear programming; i.e., he has summarized the annual production capabilities of all his factories and the annual product demands of all his warehouses and has computed that allocation of plant production to warehouse demand which would have minimized his transportation costs. In comparing the cost implied by this analysis to his total actual cost for the year he has found that if he had followed the recommendations of the model he would have saved $300,000.

   He has asked you to suggest how he might take advantage of this analysis in his future operations.

   Outline your suggestions.

4. For a man who has studied production management in some detail at M.I.T., what methods, theories and ideas which he studied would be useful to him if he took a job in long range corporate planning? How would he use them? What methods would not be particularly useful and why?

May 1965
Burns
Crowell

PRODUCTION

1.

In their book, ANALYSIS OF INDUSTRY SYSTEMS, Hadley and Whitin describe a multi-echelon inventory system and suggest that "Most inventory systems encountered in the real world are multi-echelon in nature". On the following page, however, they indicate that "The basic inventory system that will be considered in this text will be much simpler than the ... multi-echelon system ...". They go on to suggest a number of reasons for this simplification and say the most important reason is that it is very difficult to study analytically multi-echelon systems.

Consider the following multi-echelon situation:

A store room maintains stocks of 50 different items on its shelves to serve an assembly operation. The assembly operation produces 15 different finished products. Each finished product is made up of 10 stock items (parts from the store room shelves). A stock out of one or more needed items will prevent assembly.

a. Describe as clearly as you can the process by which you would determine the optimum stock out frequency for each of the fifty items. Use a model to express your thoughts, if possible.

b. Describe as clearly as you can how your recommendation of stock out frequencies would be affected by increasing product complexity (i.e. more parts per product).

2.

Discuss the problems of using digital simulation as an experimental medium in designing, say, production control systems. Illustrate your discussion with examples from the literature, if possible.

3.

The Power Plus Engineering Company manufactures a line of 12 sizes of equipment used in research in magnetic resonance of fundamental particles. These units consist of a magnetic field with its power supply as well as certain standardized test equipment. These units are sold to some 200 potential customers throughout the world. Since the units are rather expensive, a considerable period usually elapses between the creation of the need and the fulfillment (in the form of an order). This delay includes time for writing proposals, review of the proposals by funding agencies, the approval by state legislatures, etc.

PPEC works closely with prospects on a continuing basis. The representative is aware of when the board or committee will meet, what else is up for approval, etc. As a result the company is particularly well-informed about its market.

Reasonably long delivery delays are acceptable to the customer. However, this will not cover procurement and fabrication times for the more time-consuming items. Therefore PPEC inventories raw materials, components (often common to several sizes) and only the smaller sizes of complete units. Nevertheless, the company finds it necessary to forecast sales for facilities planning, scheduling and smoothing production.

During the 45 minutes that you have available for this question I would like you to develop a scheme by which the sales forecast may be constructed. This should be sufficiently detailed so that I know which information is being supplied by each person or office and how it will be combined into a finished forecast. To be specific, think in terms of forecasting the usage of a part (a ring) manufactured by PPEC and used in three mutually exclusive sizes, i.e. a customer buys only one size. Relate your forecast to the number of rings to be produced on the next run.

4.

Suppose the following 4 period production scheduling problem in which it is known that

3 units are required at the end of period 1,
5 units are required at the end of period 2,
4 units are required at the end of period 3,
6 units are required at the end of period 4.

The unit costs of producing the i^{th} item, $i = 1,\ldots,6$, in period j, $j = 1,\ldots,4$, are given in the following table,

i \ j	1 #1	2 #2	3 #1	4
1	~~$1~~	$1	$2	$1
2	~~1~~	2	2	2
3	2	2	3	3
4	2	4	3	4
5	3 4	4	4 5	5
6	3 4	4	4 5	6
r	3	5	4	6

(A) 5+3=8 so produce at least 2 in 2
MC(3) = 2 but cc of 2 = 4 = more MC(4)
so never pay to carry from 2→3

The per unit cost of carrying an item in inventory from one period to the next are the following,

from period 1 to period 2, $1 per unit,
from period 2 to period 3, $2 per unit,
from period 3 to period 4, $1 per unit.

The objective is to minimize the total production and inventory carrying costs while meeting all requirements. (Note: the units produced in any period can be used to meet the requirements in that period or any later period.)

<u>Questions</u>:

1. Find the optimal number of units to schedule in each period. If you are successful go on to part 2, otherwise go to part 3.

2. If you answered part 1 can you state a general theorem (or rule) that will allow one to solve any production scheduling problem with the same characteristics as the given problem has? When finished answering go on to part 4.

3. If you could not answer part 1 can you describe one or more methods by which you think an answer to part 1 could be found? Please give sufficient details. When finished answering go on to part 4.

4. What do you see as the pecular characteristic(s) of this problem that should be exploited in looking for a special method of solution?

OR generals

LP:

✓ (1) Use of multipliers (dual variables)
(2) Caterer problem: formulate as a std transp problem
(3) Formulate absolute error estimates as LP & describe dual
(4) Piece-wise linear objective fn: formulate as LP & dual

SDT:

(1) Faulty prodn line: formulate
(2) Conf intervals & approx of sample as Normal
✓ (2') Bayesian analysis
✓ (3) Pareto Bayesian analysis: develop
(4) 2 act problem & ENR

SP:

✓ (1) Queuing & stochastic customer characteristic
✓ (2) Burn-in testing
✓ (3) Stochastically repeating queue.
✓ (4) Advertising effectiveness.

Kenadra

Sloan School of Management

Ph.D. General Examination
Operations Research

September 14, 1966
9 - 12 a.m.

The examination is closed book. In case of doubt about question interpretation, make the most reasonable interpretation you can and note accordingly. The three questions are weighted equally.

1. A person walks to a phone booth and, finding it empty, makes a call. The call length is a random variable. While he is calling, people are arriving as a Poisson process. Let

$b(t)$ = probability density of call length, $\tilde{t}$.

$1/\mu = E(\tilde{t})$

λ = arrival rate

Each arriving person has some number of dimes in his pocket. This number is a random variable with a geometric distribution. Let

$g(k)$ = probability mass function of the number, $\tilde{k}$, of dimes in one person's pocket

$= \alpha^k(1 - \alpha)$

(a) What can you say about the number of dimes in the line when the original person emerges from the phone booth? Specifically, what is the z-transform of the number of dimes?

(b) What is the expected number of dimes in the line? If, when the person finishes his call, he wants to borrow a dime to make another call, what is the probability that there will be someone there who can lend him one?

2. The Fitrite Shoe Company has just purchased a new machine (the first of its kind). This machine, aptly called the Lacing Boxer, takes matched pairs of shoes, laces them and ties bows and then places the pairs in closed boxes for shipment. The lacing mechanism, however, is tricky to set, but once set it will never change. It is known that different settings will result in 5%, 10%, 15%, and 20% of the pairs having at least one untied shoe.

The Marketing Division has started an advertising campaign using the slogan "Fitrite the shoe that's tied to be fit for your foot." Consequently there is a great deal of management pressure to get the machine running at the 5% level as soon as possible. Outline the startup procedure you would suggest that the production department follow in introducing the Lacing Boxer into the production line.

Assume for the sake of concreteness that the machine runs at 60 pairs per minute and that finished pairs can be inspected at the rate of 10 pairs per minute per inspector.

3. Gilmore and Gomory in their paper on the cutting stock problem (<u>Opns. Res.</u>, <u>9</u>, 849) say:

"Some linear programming problems arising from combinatorial problems become intractable because of the large number of variables involved. Usually each variable represents some activity and the difficulty is that there are too many possible competing activities satisfying the combinatorial restrictions of the problem. . . . .

. . . this difficulty can be overcome . . . with the idea which is essentially this. When, in the simplex method, we reach the stage of "pricing out" or looking for a new column or activity that will improve the solution . . . we simply create a useful column by solving an auxiliary problem."

With this in mind consider the following problem:

Find $x_j \geq 0$ $(j \neq 0)$, max x_o to satisfy

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} x_o + \begin{bmatrix} 1 \\ -1 \\ 6 \end{bmatrix} x_1 + \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} x_2 + \begin{bmatrix} -4 \\ -1 \\ 1 \end{bmatrix} x_3 + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} x_4 = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}.$$

where the y_1, y_2, and y_3 may be chosen in any way satisfying

$$y_1 + 2y_2 + 3y_3 = 2, \quad y_i \geq 0.$$

A basic feasible solution for this LP is

$$x_o = -24,\ x_1 = 2,\ x_2 = 3,\ x_3 = x_4 = 0.$$

(a) What are the basic vectors for this solution? What equations determine the simplex multipliers π_1, π_2, π_3? (See note below for definition of simplex multiplier.) What are the values of the simplex multipliers?

(b) Set up an auxiliary problem to find a new column vector for introduction to the basis.

(c) Solve for the new column by inspection.

(d) After this column is introduced, some previous column will leave the basis (the one for x_2, as it turns out) and we have a new basic feasible solution. As it happens, the new solution is not optimal. What steps would you go through to find a better solution?

(e) How will you know when you have reached the optimum?

Definitional Note: For the LP:

Find $x_j \geq 0$, min. z satisfying

$$P_1x_1 + \ldots + P_nx_n = Q$$

$$c_1x_1 + \ldots + c_nx_n = z$$

where $P_j = [a_{1j} \ldots a_{mj}]^T$, $Q = [b_1 , \ldots, b_m]^T$,

let $B = [P_{j_1}, \ldots, P_{j_m}]$ = a basis

$\gamma = [c_{j_1}, \ldots, c_{j_m}]$ = corresponding cost coefficients

Then the simplex multipliers, $\pi = [\pi_1, \ldots, \pi_m]$, may be defined by

$$\pi = \gamma B^{-1}$$

or

$$\pi[P_{j_1}, \ldots, P_{j_m}] = [c_{j_1}, \ldots, c_{j_m}]$$

and the simplex optimality test becomes

$$\bar{c}_j = c_j - \pi P_j \geq 0 \quad j = 1, \ldots, m \ .$$

Sloan School of Management

Ph.D. Examination
Operations Research

May 16, 1966
9 - 12 a.m.

Instructions: Do any 3 of the 4 questions. The 3 questions you select will be weighted equally. In case of doubt about question interpretation, make the most reasonable interpretation you can and note accordingly. The examination is closed book.

* * * * *

LP

1. The Smith Widget Co. has just received a contract to supply R widgets to a customer during the next n days. The contract specifies that the Smith Co. will deliver r_j (≥ 0) widgets on the j^{th} day, j = 1, n. ($\sum_{j=1}^{n} r_j = R$)

The Smith process for manufacturing widgets involves a considerable amount of welding. The entire contents of one tank of a special gas are consumer in the welding of one widget.

Smith can buy new tanks at a cost of a dollars each and get immediate delivery. These new tanks are initially filled with the special gas. When the gas from a tank has been used, the tank can be sent back to the supplier for re-filling. The refilling service costs b dollars per tank. Tanks sent out for refilling at the end of the j^{th} day are available for use again at the beginning of the $j + p^{th}$ day.

The supplier also offers a higher cost "fast" service for refilling which returns the tanks sent out at the end of the j^{th} day at the beginning of the $j + q^{th}$ day ($q < p$ and both integers) at a cost of c dollars per tank.

Currently there are no tanks in stock or being refilled.

The Smith Co. wishes to meet the gas tank requirement for this contract at minimum cost.

(a) Formulate this problem as a linear programming problem.

(b) The Smith Co. has a computer and several programs for solving linear programming problems. In particular they have a program for solving the standard transportation problem. Show how Smith could employ this program to solve their problem, by setting up the problem as a transportation problem for the data given below. Specify all relevant supplies, requirements and costs. Do not solve the problem.

Data

$n = 7$
$p = 4$
$q = 2$
$a = 10$
$b = 3$
$c = 5$

day	required production
1	130
2	70
3	60
4	100
5	80
6	90
7	120

SP/2. Burn-In Testing. Certain types of equipment are subject to early failure. In such cases the expected remaining life of the equipment may increase with the amount of time that the equipment has run without a failure. Thus by "burning in" new units, a manufacturer can save on warranty costs or otherwise produce a better product.

Let $\tilde{t}$ = life of the equipment

$f(t)$ = p.d.f. of $\tilde{t}$

$= b\, a^b\, (a + t)^{-(b + 1)} \qquad a > 0,\ b > 1$

$\tilde{r} = \tilde{t} - \tau$ = remaining life of equipment that has lasted to τ.

(a) Find $E(\tilde{r}/\tau)$, the expected remaining life given that the equipment has lasted to τ. What is the effect of τ on expected remaining life?

A burn-in program is proposed as follows: When an order comes in for the equipment, a newly made unit is put on test and run until time T or failure, whichever comes first. If the unit fails, it is replaced by a fresh one and the process is repeated. This is done until one lasts for a time T.

Let $u(T)$ = expected warranty cost for a unit released after a burn-in time T.

c_1 = cost of fresh unit

c_2 = cost/unit time of burn-in facility

$v(T)$ = expected cost (burn-in, possible replacements, warranty) of supplying a unit to a customer

(b) Find an expression for $v(T)$ in the case of arbitrary $f(t)$.

(c) Indicate how to determine optimal T.

(d) What is the expected time to complete burn-in, given T and arbitrary $f(T)$?

SDT

3. (a) A typical confidence interval statement made by a classical statistician is:

> "on the basis of our sampling experiments, the proportion p of our customers who prefer package x over y lies between $.34 \le p \le .46$ with confidence .95".

Many businessmen would misinterpret this statement as meaning: "the probability that p falls in the interval .34 to .46 is .95."

(i) Why is the businessman's statement incorrect?

(ii) If you are a Bayesian and regard p as a random variable, under what conditions can a confidence interval statement like that just given be interpreted as a statement about a posterior probability interval? (For purposes of discussion assume that sampling is Bernouilli with parameter p and you have observed a sample of size n. Of the n responses, r prefer x over y.) Be explicit.

(b) You are analyzing a decision problem in which the mean of a population is of central concern. You know neither the population mean μ nor the population variance ν with certainty, and you know that the population is not normal. You have observed a sample $x_1, \ldots, x_n$ of independent observations from the population.

(i) State the conditions under which the following procedure gives a reasonable approximation to the posterior density $f''(\mu)$ of the population mean $\tilde{\mu}$ when you have assigned a prior $f'(\mu)$ to μ:

(1) Compute the sample mean

$$m = \frac{1}{n} \Sigma x_i$$

and an unbiased estimate

$$v = \frac{1}{n-1} \Sigma (x_i - m)^2$$

of ν. Assume $\nu = v$. n is large

(2) Regard the likelihood of m given μ and $\tilde{\nu} = v$ as normal with mean μ, variance $\tilde{\nu}/n = v/n$; i.e. $\tilde{m} \mid \mu, v$ has density $f_N(m \mid \mu, v/n)$.

(3) Compute the approximate posterior density of $\tilde{\mu}$ as $f''(\mu) \propto f_N(m \mid \mu, v/n)\ f'(\mu)$.

(ii) Under what conditions might treating v as if it were the true value of $\tilde{\nu}$ lead to serious errors?

(iii) Give a rationale for the conditions you stated in (i) leading to a reasonable approximation of $f''(\mu)$ in instances other than those characterized by (ii).

(c) (i) Consider a two act decision problem in which the cost or value of each act is a linear function of the mean μ of a population. You have assigned a prior $f'(\mu)$ to μ. Explain the role that the prior distribution of the posterior mean plays in the process of determining whether or not you should take a fixed size sample before choosing one of the two acts, and if so, how large the sample should be. Pay particular attention to the concepts of <u>expected value of perfect information</u> (EVPI), <u>expected value of sample information</u> (EVSI), and <u>expected net gain</u>.

(ii) Under what circumstances does EVPI = EVSI?

SDT

4. Suppose that the set of events $B = \{B_i : i = 1,\ldots,n\}$ form a partition of the sample space Ω, i.e., the events in B are mutually exclusive and collectively exhaustive. Let $f'(B_i)>0$, all $B_i \in B$ denote a mass function for the events in B.

a. For any event $A \in \Omega$ show that

$$(1) \qquad P(A) = \sum_{i=1}^{n} P(A|B_i)\, f'(B_i) \qquad \sum_{i=1}^{n} P(A, B_i)$$

(Equation (1) is sometimes called the Theorem of Total Probabilities)

b. Let $f''(B_i|A)$, $B_i \in B$ denote the (posterior) mass function for the events in B given that an event $A \in \Omega$ has occurred. Show that $f''(B_i|A)$ is related to $f'(B_i)$ by the equation

$$f''(B_i|A) = \frac{P(A|B_i)\, f(B_i)}{P(A)}$$

$$(2) \qquad f''(B_i|A) = KP(A|B_i)\, f'(B_i)$$

$$K = \frac{1}{P(A)}$$

Σ ✓

where K is a normalizing constant such that

$$\sum_{i=1}^{n} f''(B_i|A) = 1.$$

c. Suppose $B = \{B_1, B_2\}$. Further suppose that a certain experiment can lead to events A_1 or A_2 and that

$P(A_1\|B_1) = 1/4$	$P(A_2\|B_1) = 3/4$
$P(A_1\|B_2) = 3/4$	$P(A_2\|B_2) = 1/4$.

The decision maker is contemplating taking one of two actions D_1 or D_2. The return function r for these actions and "states of nature" is given by

$r(D_1,B_1) = 4$	$r(D_1,B_2) = 0$
$r(D_2,B_1) = 2$	$r(D_2,B_2) = 2$

If the experiment is run and A_1 is observed, then at what value of P ($f'(B_1)=P$) will the decision maker be indifferent between D_1 and D_2. (Assume expected value is a valid criterion.)

$$f''(B_i|A_1) = \frac{1}{P(A_1)} \, P(A_1|B_i) f'(B_i)$$

$$P(A_1) = P(A_1|B_1)\,P + P(A_1|B_2)(1-P) = \tfrac{1}{4}P + \tfrac{3}{4}(1-P) = \tfrac{3}{4} - \tfrac{1}{2}P$$

$$f''(B_1|A_1) = \frac{\frac{1}{4}P}{\frac{3}{4}-\frac{1}{2}P} = \frac{P}{3-2P}$$

$$f''(B_2|A_1) = \frac{\frac{3}{4}(1-P)}{\frac{3}{4}-\frac{1}{2}P} = \frac{3-3P}{3-2P}$$

$$\left.\begin{aligned} E''(r|D_1) &= 4\frac{P}{3-2P} \\ E(r|D_2) &= 2 \end{aligned}\right\} \quad \begin{aligned} 2P &= 3-2P \\ P &= 3/4 \end{aligned}$$

Johnson
Gorry

Doctoral Examination

Sloan School of Management

Operations Research

February 9, 1966
9 - 12 AM

(closed book)

Please do each question in a different exam book.

SP 1. Consider a single server queue with Poisson arrivals of rate λ and exponential service of rate μ, with the following variation: whenever a service is completed, a departure occurs only with probability α. With probability $1 - \alpha$, the customer, instead of leaving, joins the end of the queue. Thus it is possible for the same individual to be serviced many times. Aside from this variation, assume FIFO service.

(a) Set up and solve equations for the steady-state probabilities.

(b) Find the expected waiting time of a customer from the time he arrives until he has completed service for the first time.

(c) Find the expected waiting time of a customer in the system.

(d) Find the distribution of the waiting time described in (b).

LP 2.a) Consider the linear hypothesis

(1) $\tilde{Y} = \alpha + \beta X + \tilde{u}$

where $\tilde{Y}$ and $\tilde{u}$ are random variables, α and β are unknown constants and X is an exogenous variable. Suppose n observations $\{(X_j, Y_j): j = 1,2,\ldots,n\}$

have been made. It may be assumed that $X_i \neq X_j$, $i \neq j$.

The usual approach to finding estimates for α and β is to assume

$$E(\tilde{u}_i) = 0$$

$$E(\tilde{u}_i \tilde{u}_j) = \begin{cases} \sigma_u^2, & i = j,\ j = 1,\ldots,n \\ 0, & i \neq j,\ j = 1,\ldots,n, \end{cases}$$

and to perform a least squares analysis. Instead of this approach suppose that the objective is to find estimates for α and β, say a and b, such that the sum of the absolute values of the vertical deviations of the points from the fitted line is minimized. Formulate this problem as a linear programming problem with n constraints.

good

b) Consider a linear program with m constraints and n unknowns where the constraint set is partitioned into two mutually exclusive and collectively exhaustive classes M_1 and M_2 and similarly the variables into two such classes N_1 and N_2. The linear program is written as follows:

(2) Maximize $\sum_{j=1}^{n} c_j x_j$

subject to the constraints

(3) $$\sum_{j=1}^{n} a_{hj} x_j \begin{cases} \leq b_h, & h \in M_1 \\ = b_h, & h \in M_2 \end{cases}$$

(4) $$x_j \begin{cases} \text{non-negative for each } j \in N_1 \\ \text{unrestricted in sign for each } j \in N_2 \end{cases}$$

Describe the dual linear program corresponding to (2), (3) and (4).

hm.

c) Using the result of part b write the dual program to your solution in part a. Let $\{d_j: j = 1,\ldots,n\}$ denote the dual variables.

d) Let $f_j = d_j + 1$, $j = 1,\ldots,n$ in part c. After making this substitution describe the resulting linear program as compared with the program in part c.

SDI 3. A random process of considerable interest to economists is the Pareto process. The density function of a Pareto random variable is

$$f(x|\alpha, x_o) = \begin{cases} 0 & \text{if } x \le x_o \\ \dfrac{\alpha x_o^{\alpha}}{x^{\alpha+1}} & \phantom{\text{if }} x > x_o \end{cases}, \quad \begin{matrix} \alpha > 0 \\ x_o > 0 \end{matrix} \quad . \tag{1}$$

This density function, however, possesses properties that makes some aspects of Bayesian inference difficult. Your task is to develop a Bayesian view of this process.

The likelihood of observing values $x_1, \ldots, x_n$ of n random variables that are mutually independent and identically distributed according to (1), given α and x_o, is

$$(\alpha x_o^{\alpha})^n \prod_{i=1}^{n} (x_i)^{-(\alpha + 1)} . \tag{2}$$

(i) Assuming the x_o is known with certainty but that α is not, define a sufficient statistic. (Hint: n and one other statistic, call it t).

(ii) Letting $\lambda = \log_e(t/x_o^n)$, find the kernel of the likelihood of (2) in terms of α, and λ, and identify a natural conjugate family of priors for $\tilde{\alpha}$ considered as a random variable.

(iii) Assuming that you have assigned a natural conjugate prior to $\tilde{\alpha}$ and observed a set of sample observations leading to the sufficient statistic (n, t), show how to go from the prior to a posterior on $\tilde{\alpha}$. Do this explicitly in terms of parameters of the prior and (n, t). (Stating Bayes Theorem is not enough!)

(iv) What difficulties arise if you wish to use a natural conjugate prior on $\tilde{\alpha}$ when faced with a decision problem under uncertainty for which the mean of the Pareto process is the underlying parameter of interest?

Ph.D. Examination
September 20, 1965

Operations Research

The three questions are weighted equally. In case of doubt about the interpretation of a question, interpret it in the most reasonable way you can and note accordingly.

LP

1. A smelting company has N types of ore that it can buy. These are purified into M different rare earth elements for use in phosphors. Let

x_j = purchase rate of j^{th} ore. (kg/mo.)

e_{ij} = efficiency of recovering i^{th} element from j^{th} ore (kg of i/kg of j)

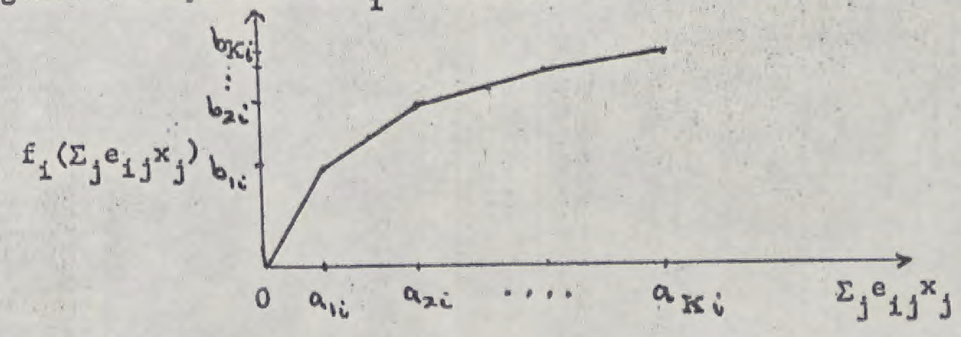
$f_i(\Sigma_j e_{ij} x_j)$ = profit on the production of element i, exclusive of purchase costs. (dol./mo.)

P_i = minimum production of element i required by contractual commitments. (kg/mo.)

c_j = price of ore j. (dol./kg)

d = available budget for purchases. (dol./mo.)

The general shape of the $f_i(\cdot)$ is piecewise linear as follows:



a) Set up a linear program for determining the purchase rates of each ore.

b) Write down a dual linear program and interpret its variables.

c) Suppose that the ores are subject to quantity discounts: The price is c_{1j} for the first 10,000 kg, $c_{2j} < c_{1j}$ for the second 10,000 kg, $c_{3j} < c_{2j}$ for the third etc. What difficulties does this introduce if any?

Ph.D. Examination
Operations Research

SP 2. An airline is advertising in magazines to promote vacations in the Caribbean. Let

$i = 1, 2, \ldots, k$ index the magazines

a_i = probability a person subscribes to magazine i

z_i = probability that a subscriber to i sees an ad in i.

n_i = number of issues of i in which ads are placed.

$A(1-b^r)$ = probability that a person who sees r ads takes a Caribbean trip.

The probability that a subscriber to i sees an ad in an issue is independent of whether he sees an ad in other issues of i. The probability that a person subscribes to i is independent of whether or not he subscribes to other magazines.

What is the probability a person takes a Caribbean trip?

$$P(T) = \sum_{r=0}^{\infty} P(T|r)\,P(r) = \sum A(1-b^r)\,P(r)$$

$$= A - P^T(b)$$

x_i = x ads seen in magazine i

$$\therefore r = \sum_{i=1}^{x} x_i$$

$$\sum_{r=0}^{\infty} A(1-b^r) = 1$$

$$A - \sum_{r=0}^{\infty} b^r$$

A

$$\therefore P_r^T(b) = \left[P_{x_1}^T(b)\right]\left[P_{x_2}^T(b)\right] \cdots$$

$$P(x_i = X) = a_i \binom{n_i}{X} z_i^X (1-z_i)^{n_i - X}$$

$$P_{x_i}^T(b) = a_i \sum_{X=0}^{n_i} \binom{n_i}{X} (z_i b)^X (1-z_i)^{n_i - X} = a_i\left(z_i b + (1-z_i)\right)^{n_i}$$

$$\therefore P(T) = A - P_r^T(z) = -\prod_{i=1}^{k} a_i\left[z_i b + (1-z_i)\right]^{n_i} + A$$

Ph.D. Examination
Operations Research

SDT

3. There is a list of N^* potential customers for a product. Each of these customers can only buy one unit of the product. If any customer is contacted there is an unknown probability $\tilde{p}$ that he will purchase the product. The customers can be contacted in two stages, a sampling stage and a main stage. If a sample of n is taken, the decision whether or not to contact the remainder $(N^* - n)$ of the customers will be based on the number of affirmitive answers in the sample.

In the sampling stage let the cost of contacting n customers be $(K_s + c_s n)$ and let the revenue be π_s per success.

In the main stage let the cost of contacting $(N^* - n)$ customers be $[K_t + c_t(N^* - n)]$ and let the revenue be π_t per success.

Let $\tilde{p}$ be given a beta density with parameter (r', n').

NOTE: A beta density with parameter (r', n') is

$$\frac{\Gamma(r')\,\Gamma(n' - r')}{\Gamma(n')}\; p^{r'-1}(1 - p)^{n'-r'-1} \qquad \begin{array}{l} 0 < p < 1 \\ r' > 0 \\ n' \geq r' \end{array}$$

And the function

$$L^{(r)}_{\bar{p}''}(x) \equiv \int_x^{\infty} [\,\bar{p}'' - x]\, d\,F(\bar{p}'')$$

where $F(\cdot)$ is the cumulative distribution function of $\bar{p}''$ prior to observing the value of r when a sample of size n is to be taken.

a) Show that if a sample of size n is taken and r successes obtained then the remainder of the population should be contacted if and only if

$$\bar{p}'' > v(n)$$

where

$$\bar{p}'' \equiv \frac{r' + r}{n' + n} \qquad \text{and} \qquad v(n) \equiv \frac{K_t + c_t\,(N^* - n)}{\pi_t\,(N8 - n)}$$

b) Let $V(n, r)$ denote the expected revenue if a sample of size n is taken, r successes are observed and an optimal policy is then followed. Show that

$$V(n, r) = -K_s - n\,c_s + r\,\pi_s + \Big\{-K_t + (N^* - n)\,[-c_t + \bar{p}''\,\pi_t]\Big\}^+$$

where

$$\{x\}^+ \equiv \max(x, 0).$$

3. Continued

c) If n is determined but r is still unknown then the expected net revenue (ENR) is defined as

$$V^*(n) \equiv E_r\, V(n, r).$$

Show that

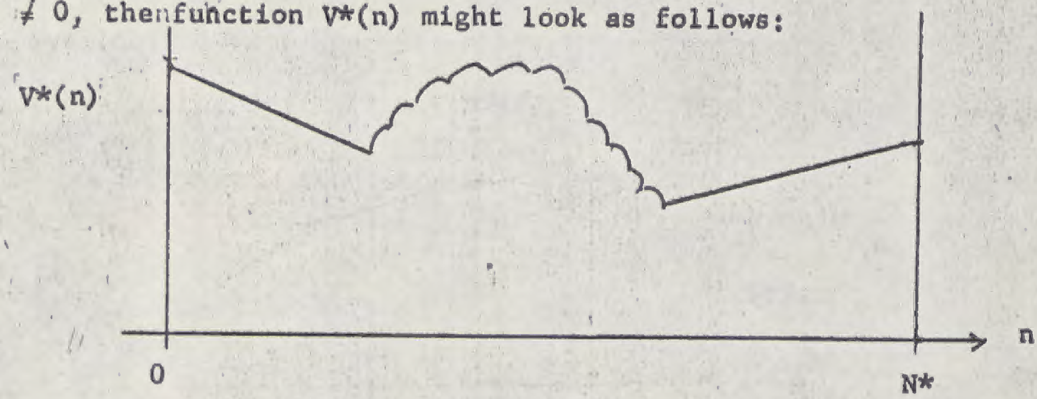
$$V^*(n) = -K_s + n(\bar{p}'\pi_s - c_s) + (N^* - n)\,\pi_t\, E\left\{\tilde{\bar{p}}'' - v(n)\right\}^+$$

[Note. A digital computer program is available which computes $V^*(n)$ recursively for n = 1, 2, ... and about 1000 values of n can be computed in a single minute.]

d) If $K_t = 0$, then show that

$$V^*(n) = -K_s + n(\bar{p}'\pi_s - c_s) + (N^* - n)\,\pi_t\, L^{(r)}_{\bar{p}''}(c_t/\pi_t).$$

e) When $K_t \neq 0$, thenfunction $V^*(n)$ might look as follows:



Explain the linear behavior when n is small, the scalloping behavior for intermediate values of n, and the linear behavior for large values of n.