6.534 T

GAMES AND DECISIONS

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6.534T : Games and Decisions

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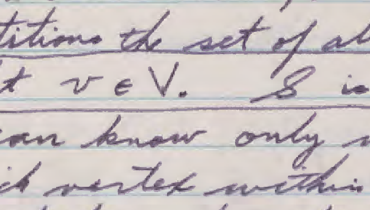
Reference: Blackwell & Girshick, Theory of Games & Statistical Decisions

The "moves" in a game are the actions taken by the "players" participating or the (probabilistic) occurrence of events which change the relationship of the players. Moves can be classed as "personal" or "chance." For any given move, the rules of the game must specify:

Personal moves	Chance moves
1. Who is to move?	1. What are the alternatives?
2. What are his choices?	2. What is the probability distribution of the alternatives?
3. What information does he have about previous choices & outcomes?	

The state of the game is specified by a set of vertices $\{v\}$. which are choice moves, and a set $\{w\}$ which are chance moves. $\mathcal{S}$ partitions the set of all vertices into information sets V, such that $v \in V$. $\mathcal{S}$ is the "information partition." The player can know only what information set he is in, and not which vertex within the set. This partition of vertices ($\mathcal{S}$) is specified by the rules of the games. From this, it follows that a player can not know exactly what move was made by the previous player & can not distinguish between the alternatives at each of the vertices in the set. (I.e., the alternatives at each vertex in an information set appear identical to the player.

Game of perfect information $\equiv$ all V contain only one vertex.



"O" indicates chance move

I → indicates player I

Information sets are the vertices enclosed by one boundary.

Personal choice moves:

z = information available to player about past moves

$z \in Z$ where $Z = \{$all possible z's$\}$

s = players choice of alternatives available to him

$s \in S$ where $S = \{$all alternatives available to the player$\}$

$S(z) = \{$all alternatives specified by z, i.e., at a particular information set $z\}$

The players strategy is ~~determined (or specified)~~ specified by a listing of which branch leading out of each information set ~~at~~ will be selected. That is, a listing of what move he will make in ~~any~~ all possible situations.

In formal terms, the players strategy is determined by a function $f(\cdot) \ni$

$f(z) \in S(z)$ for all z in Z is the strategy.

$f(z)$ is a rule which specifies, for each information set, which alternative will be selected.

$f \in F = \{$all possible strategies for a given player$\}$

Chance moves:

$w \in W$ where $W = \{$all possible ~~circumstances~~ vertices in which a chance move is required.$\}$

$S(w) = \{$all possible alternatives for the particular $w\}$

There is a probability distribution P over $S(w)$ according to which a selection s is made; $s \in S(w)$.

~~Suppose~~ Let $h(w)$ specify a particular selection s from $S(w)$ which has been made in accordance with the P covering $S(w)$. A selection for each w constitutes a function $h(\cdot) \ni h(w) \in S(w)$ and $\ni$ $h(\cdot)$ is defined on W.

Each h is a specific outcome of an "overall" trial in which one outcome is found for each chance vertex. h says, "If the game reaches vertex 97, which is a chance move, the choice will be A from alternatives A, B, C."
$h \in \mathcal{H}$ where $\mathcal{H} = \{$all possible sequences of the chance moves in a game$\}$.

The distributions P on each chance move determine a distribution P on $\mathcal{H}$. Picking h according to P is the same as choosing an s from $S(w)$ according to the relevant distribution P.

To play the game, we can simply select a strategy for each player, an h according to P, and follow the game through to the end; these quantities, once chosen, completely specify the outcome of the game (uniquely).

Assume there are K players. Let

f_i = a possible strategy for player i.

$f_i \in F_i = \{$all possible strategies for player $i\}$

The ~~outcome~~ course of play is completely specified by the $(K+1)$-tuple $g = (f_1, f_2, \ldots, f_K, h)$.

$g \in G, G = \{$all possible courses of play$\}$

Now, for each course of play there is an outcome $r \in R$

$R = \{$all possible outcomes of the game$\}$.

Since for each g there is a unique r, we can define a function $\mathcal{g}$ which is defined on $G \ni G$ is mapped <u>onto</u> R

$\mathcal{g} : G \rightarrow R$,

For fixed f [f is the k-tuple $(f_1, f_2, \dots, f_k)$], the distribution P over $\mathcal{H}$ determines a distribution Q_f over G which in turn determines (via φ) a distribution π_f on R.

The game is now simply played as follows. Each player i chooses a strategy f_i from F_i. This fixes $f = (f_1, \dots, f_k)$ and π_f. The result of the game is then chosen from π_f. The result of the game can be specified only up to a probability function.

<u>The utility function</u>:

So far, we have said nothing about the preferences a player may have for certain outcomes. As mentioned above, we can specify the results only up to a probability distribution. Let each player rank the possible probability distributions over R in order of his preference. <u>This ranking is called the player's preference pattern or preference rating.</u>

Now, let $u(\cdot)$ be a function defined over $R \ni u(r_1) > u(r_2)$ if & only if r_1 is preferable to r_2 when the $r \in R$ have been ranked by the player in order of preference.

A prob. dist. $\xi^{(1)}$ over R is preferable to $\xi^{(2)}$ if & only if the function

$$\boxed{U(\xi) = \sum_{r \in R} \xi(r)\, u(r) = \sum_{j=1}^{N} \xi_j\, u(r_j)}$$

is such that $U(\xi^{(1)}) > U(\xi^{(2)})$.

$$\boxed{U_i(\cdot) \equiv \text{player } I\text{'s utility function}}$$

Now, for any given strategy f, and the associated π_f, we define

$$\boxed{M_i(f_1, \ldots, f_K) = \mathcal{U}_i(\pi_f)}$$

M_i & u_i & $\mathcal{U}_i$ are numerical fns.

We now characterize the game by the sets ~~triple G =~~ $X, Y, M_1, M_2, \ldots, M_K$ and assume that it is the aim of player i to maximize M_i.

Two-person zero-sum games:

From here on, we restrict ourselves to the situation where:

$$K = 2 \quad ; \quad \sum_i M_i(f_1, f_2) = 0 = M_1(f_1, f_2) + M_2(f_1, f_2).$$

Since what player I wins, II loses, there will be no collusion for mutual benefit. Also, since $M_2 = -M_1$, we need specify only M_1. Hence, we can express the game in the normal form by the triple $G = (X, Y, M)$. X & Y are the strategy spaces for players I & II, and M is a bounded numerical function of the pair (x, y), $x \in X$, $y \in Y$ which is payed to I by II.

Reduction: $G' \, r \, G$, i.e., G' is a reduction of G $\Leftrightarrow$

(a) $\exists\, f : X$ onto $X' \ni M(x,y) = M'(f(x), y)$ for all $y \in Y = Y'$

or

(b) $\exists\, g : Y$ onto $Y' \ni M(x,y) = M'(x, g(y))$ for all $x \in X = X'$.

Equivalence: $G' \sim G$, i.e., G' is equivalent to G $\Leftrightarrow$

$\exists\; G_1, G_2, \ldots, G_N \ni$

$$\left.\begin{array}{l} G' r G_1 r G_2 r \ldots r G_N r G \\ \text{or } G r G_N \ldots r G_2 r G_1 r G \end{array}\right\} \quad G' \sim G \Leftrightarrow G \sim G'$$

Solutions to a game:

We now try to find a way for a player to select strategies so as to maximize his return. We assume a finite game with two highly intelligent players.

We can characterize the payoff M by a matrix $A = \| a_{ih} \|$ $\ni$ if the strategies chosen are x_i and y_h, the payoff to I by II is a_{ih}. A is $j \times k$

$$\begin{array}{cc} & \text{II} \\ \text{I} & \left\| \begin{matrix} a_{11} & a_{21} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \cdot & \cdot & - & - \\ a_{j1} & a_{j2} & \cdots & a_{jk} \end{matrix} \right\| \end{array}$$

For a given strategy ~~x~~ i ~~for~~ chosen by player I, the maximum return he can be certain of is

$$\min_h a_{ih}$$

By varying his strategy, he can approach the maximum payoff which he can guarantee:

$$\boxed{\max_i \min_h a_{ih} \equiv \lambda_G^*}$$

Or, in terms of the sets X, Y, $\lambda_G^* = \sup_{x \in X} \inf_{y \in Y} M(x,y)$

Similarly for player II, who wants to minimize the payoff: The maximum II need pay I is:

$$\boxed{\min_h \max_i a_{ih} \equiv v_G^*}$$

Or, $v_G^* = \inf_{y \in Y} \sup_{x \in X} M(x,y)$

If $\lambda_G^* = v_G^* = v_g$, the players will both choose a strategy that yields this payoff and will not try to change strategies. If v_g exists, a player <u>cannot</u> improve his payoff by an occasional "raid" strategy if his opponent plays the "optimal" strategy, as all alternatives give a less favorable payoff. For this reason, neither player will deviate from the strategies which realize v_g. These are called <u>good strategies</u> and v_g is called the <u>value of the game</u>. v_g is a "saddle point" of the payoff matrix, and play which realizes this is called <u>optimal play</u>.

<u>E.g.:</u>

$$A = \begin{bmatrix} -5 & 3 & 1 & 20 \\ 5 & 5 & 4 & 6 \\ -4 & -2 & 0 & -5 \end{bmatrix} \qquad \left.\begin{array}{l} \lambda_G^* = a_{23} = 4 \\ v_G^* = a_{23} = 4 \end{array}\right\} v_g = 4.$$

This game has a pure value & the players "best" strategies are well defined.

There need not be a saddle point, e.g., penny matching:

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad \left.\begin{array}{l} \lambda_G^* = -1 \\ v_G^* = 1 \end{array}\right\} v_g \text{ is not defined.}$$

Here, it ~~s~~ is clear that we cannot specify an optimal play in terms of the two pure strategies available to each player.

An aside: supremum and infimum:

Let $S = \{s : s \text{ is a real number}\}$; if there exists an upper bound y such that $y \geq s$ for all $s \in S$, ~~the~~ where y need not belong to S, then S has an upper bound. The least upper bound x is the smallest number that is an upper bound; i.e., given $\epsilon > 0$, $x - \epsilon$ is not an upper bound for arbitrarily small ϵ. This least upper bound of a set S is defined as

$$\sup S.$$

The least upper bound of a function ~~f~~ on S is

$$\sup_{s \in S} f(s), \ s \in S \quad = \text{least upper bound on set } B \text{ of points } b = f(s),\ b \in B.$$

Similarly, we can define a greatest lower bound to the set S, or $\inf S$. Also,

$$\inf_{s \in S} f(s) = \text{greatest lower bound of points } f(s) = b \in B.$$

Max and min:

If $\sup_{s \in S} f(s) \in B$, then $\sup_{s \in S} f(s) = \max_{s \in S} f(s)$

and similarly, if $\inf_{s \in S} f(s) \in B$ then $\inf_{s \in S} f(s) = \min_{s \in S} f(s)$

Note that if B is finite, $\sup f(s)$ and $\inf f(s) \in B$ and we can use the simpler functions max and min.

Properties of inf and sup:

$$\sup (f+g) \leq \sup f + \sup g$$

$$\inf (f+g) \geq \inf f + \inf g$$

Solution + pure values of infinite games:

Consider $G = (X, Y, M)$ where X and Y may be infinite. $M(x,y)$ describes all possible outcomes of a strategy x picked by player I from X. For his chosen x, the minimum player I can guarantee is

$$\boxed{\inf_{y \in Y} M(x,y) \equiv \Lambda_G(x)}$$

This function $\Lambda_G(x)$ now describes player I's security level for each strategy $x \in X$ available to him. He will want to find the particular strategy which maximizes this quantity so that he can guarantee himself as much as possible. This quantity is

$$\boxed{\sup_{x \in X} \Lambda_G(x) = \sup_X \inf_Y M(x,y) \equiv \lambda_G^*}$$

This is the greatest lower bound to the winnings of player I. That is, for any $\epsilon > 0$, I can guarantee $\lambda_G^* - \epsilon$ but not $\lambda_G^* + \epsilon$. This is completely independent of the strategies II may chose.

Similarly for player II & his point of view, for a chosen $y \in Y$, player II can assure that I will win no more than

$$\boxed{\sup_X M(x,y) \equiv \Upsilon_G(y)}$$

By choosing the proper $y \in Y$, II can assure that I will win no more than

$$\boxed{\inf_Y \Upsilon_G(y) = \inf_Y \sup_X M(x,y) \equiv v_G^*}$$

From these definitions, it is obvious that

$$\boxed{\lambda_G^* \leq \Lambda_G(x) \leq M(x,y) \leq \Upsilon_G(y) \leq v_G^*}$$

which implies that $\boxed{\sup_X \inf_Y M(x,y) \leq \inf_Y \sup_X M(x,y)}$

If $\lambda^*_G = U^*_G$, the game has a pure value which we call $v_g = \lambda^*_G = U^*_G$. A good strategy for either player is one in which the player tries to realize $M(x,y) = v_g$.

If $G \sim G'$, then $\left\{\begin{matrix} \lambda^*_G = \lambda^*_{G'} \\ U^*_G = U^*_{G'} \end{matrix}\right\}$:

To prove this, we need establish it for the case $G \,r\, G'$. If $G \,r\, G'$, then

$$M(x,y) = M'(f(x), y) \quad \text{for all } y \in Y$$

and hence,

$$\Lambda_G(x) = \Lambda_{G'}(f(x))$$

Since $M(x,y)$ varies over the same range as $M(f(x),y)$ for all $y \in Y$, we must have

$$\boxed{\lambda^*_G = \sup_X \Lambda_G(x) = \sup_X \Lambda_G(f(x)) = \sup_{f(x)\in B} \Lambda_G(f(x)) = \sup_{X'} \Lambda_{G'}(x) = \lambda^*_{G'}}$$

<u>Perfect information games</u>:

<u>Def</u>: A game $G = (X, Y, M)$ is a perfect information game of order zero if $M(x,y)$ is constant. G is a perfect information game of order $n+1$ if & only if there exists a class of games $G_a = (X_a, Y_a, M_a) \leftrightarrow a \in A$ ~~~~(where G_a is a ~~~~ perfect information game of order n) such that either:

(1) I moves first and $X = \{(a, z)\}$ where $z \in X_a$ and $Y = \{$all functions f defined on $A \ni f(a) \in Y_a\}$ $\Big\}$ $M[(a,z), f(a)] = M_a[z, f(a)]$

or (2) II moves first and the roles of X and Y are reversed in (1) $\Big\}$ $M[f(a),(a,z)] = M_a[f(a), z]$

or (3) the first move is random and $X = \{\text{all } x(a) \in X_a\}$ and $Y = \{\text{all } y(a) \in Y_a\}$ and $M(x,y) = \sum_{a \in A} P(a)\, M[x(a), y(a)]$

We can start at each outcome of a perfect information game and work back through the tree since each node is distinct. This will show that there must be a saddle point, so that there are good pure strategies ~~~~ & a <u>perfect information game has a pure value</u>.

Games with no pure value:

Assume we assign a value v to a finite game $G = (X, Y, M)$ If we modify G to be G_1 such that I selects a strategy $x \in X$ and then tells II which he picked. II then selects $y \in Y$ which determines $M(x, y)$. Obviously, this places I at a relative disadvantage in G_1 so $v_{G_1} \leq v_G$.

If $x_0 \in X$ is a good strategy for I, II will choose $y_0 \in Y$ such that $M(x_0, y_0) = \min_y M(x_0, y) = \lambda_G^* = v_{G_1}$.

If we modify $G \to G_2$ such that II picks $y \in Y$ and tells I before I chooses his $x \in X$. Obviously $v_{G_2} = U_G^*$

Thus, if a value v for G exists, $v_{G_1} \leq v \leq v_{G_2}$ or $\lambda_G^* \leq v \leq U_G^*$. We can think of $\Delta \equiv U_G^* - \lambda_G^*$ as the value of knowing the opponent's intention ($\Delta \geq 0$).

This shows that the value of the game depends on knowing the opponents intentions if the game does not have a pure value. Since our opponent is presumed intelligent, we should play so that he cannot deduce our intentions.

One way of doing this is to adopt a probabilistic selection of strategies. If we can find distributions over X and Y such that neither player cares if the other knows the distribution being used, then we will have found a "value" for the game. This is a "value" in the sense that neither player will find it beneficial to change if the other "plays it straight." The theory must be ~~be~~ developed on the basis that both players know the theory and play intelligently. This leads us to the concept of mixed strategies:

Mixed extension and mixed strategies:

Suppose I chooses his strategies from a distribution ξ over X such that $Pr\{x_i\} = \xi_i$. Similarly, let II select $y \in Y$ according to η over Y.

Let $Z = \{$all ξ over $X\}$ and $H = \{$all η over $Y\}$. We can then define a new game Γ as the mixed extension of G to be

$$\Gamma = (Z, H, M)$$

where $M = M(\xi, \eta) = \sum_{ij} \xi_i \eta_j M(x_i, y_j)$

~~A policy of selecting a ξ or η is called~~

A selected distribution ($\xi \in Z$ or $\eta \in H$) is called a mixed strategy. A special case of a mixed strategy is $\{\xi_k = 1; \xi_i = 0, i \neq k\}$ which reduces to ~~the~~ the pure strategy x_k.

Value of a mixed extension

Define $\Lambda_\Gamma(\xi) \equiv \inf_H M(\xi, \eta)$. It can be shown that further $\Lambda_\Gamma(\xi) = \inf_Y M(\xi, y)$. This says that II would always select a pure strategy if he knew ξ. This is self-evident as one column of the payoff matrix will always have a minimum mean for a given ξ over the rows.

Further,

$$\boxed{\sup_Z \Lambda_\Gamma(\xi) \equiv \lambda^*_\Gamma}$$

and similarly, $\sup_Z M(\xi, \eta) = \Upsilon_\Gamma(\eta)$ and

$$\boxed{\inf_H \Upsilon_\Gamma(\eta) = v^*_\Gamma}$$

If we can find strategies ξ and η such that $\lambda^*_\Gamma = v^*_\Gamma$, we will consider that we have found a "value" for the game $v = \lambda^*_\Gamma = v^*_\Gamma$. Strategies ξ and η which realize this value are good strategies in the broader sense of the word.

<u>Theorem</u>: For a finite game $G = (X, Y, M)$, where $X = (x_1, \dots, x_m)$ and $Y = (y_1, \dots, y_n)$, the mixed extension of G $\Gamma = (Z, H, M)$ has a value v_Γ such that $v_\Gamma = \lambda_\Gamma^* = U_\Gamma^*$.

<u>Proof</u>:

See Luce & Raiffa p. 391; this is Nash's proof of the Minimax Theorem:

Let $T(\xi, \eta) = (\xi', \eta')$ define a transformation T on Z and H such that $\xi' \in Z$ and $\eta' \in H$. The theorem will consist of two parts:

(1) $T(\xi, \eta) = (\xi, \eta) \Leftrightarrow \xi$ and η are good strategies

(2) T always has at least one fixed point $(\xi, \eta) \ni T(\xi, \eta) = (\xi, \eta)$.

Define transforms

$$c_i(\xi, \eta) = \begin{cases} M(x_i, \eta) - M(\xi, \eta) & \text{if } \geq 0 \\ 0, & \text{otherwise} \end{cases} \qquad i = 1, \dots, m$$

$$d_j(\xi, \eta) = \begin{cases} M(\xi, \eta) - M(\xi, y_j) & \text{if } \geq 0 \\ 0, & \text{otherwise} \end{cases} \qquad j = 1, \dots, n$$

so that

$$\xi_i' = \frac{\xi_i + c_i(\xi, \eta)}{1 + \sum_i c_i(\xi, \eta)} \quad \text{and} \quad \eta_j' = \frac{\eta_j + d_j(\xi, \eta)}{1 + \sum_j d_j(\xi, \eta)}$$

Note that for good ξ and η, $c_i = 0$ and $d_j = 0$.

Now if $T(\xi, \eta) = (\xi, \eta)$ then since $M(\xi, \eta) = \sum_i \xi_i M(x_i, \eta)$ and $\xi_i \geq 0$, we have that

$$M(\xi, \eta) < M(x_i, \eta) \text{ cannot hold for all } i.$$

Thus, at least one i is such that $M(\xi, \eta) \geq M(x_i, \eta)$ and at least one $c_i = 0$; say $c_{i_0} = 0$. Then

$$\xi'_{i_0} = \frac{\xi_{i_0}}{1 + \sum_i c_i} = \xi_{i_0} \Rightarrow \sum_i c_i = 0 \quad \& \quad c_i \geq 0 \text{ for all } i.$$

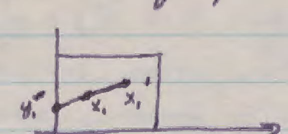
Therefore $c_i = 0$ for all i. Similarly for η, $d_j = 0$ for all j. Thus ξ is better against η than any x and η is better against ξ than any y. ~~But by the discussion on p12 of the value of Γ, we see that for a given ξ,~~

Thus ξ & η are good strategies. It is obvious from the definitions that if ξ & η are good, $T(\xi, \eta) = (\xi, \eta)$.

Now we prove that T has at least one fixed point and we have proven the existence of good strategies ξ and η.

Every pair of strategies can be represented as $(\xi_1, \dots, \xi_m, \eta_1, \dots, \eta_n) \in S_{m+n}$. Going to the special case $m = n = 2$, since $\Sigma \xi = \Sigma \eta = 1$, we can specify ξ & η by $x = (\xi_1, \eta_1)$

Let $T(x) = x'$ and assume $x' \neq x$.
Define a ~~map~~ mapping $U(x) = y \ni$ a line from x_i' to x_i hits y_i on the boundary of the unit square:



Thus we have mapped x into the boundary (& possibly onto) and none of the boundary points move under the transformation U. This is obviously a continuous map (i.e., U) but to achieve the mapping into the boundary, we must "tear" the region of x's and hence, U is not continuous. This is a contradiction, so we must have a fixed point. Hence there exist good strategies and Γ has a value.

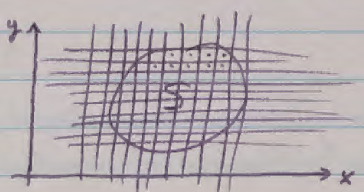
Infinite games with a value:

Theorem: If $G = (X, Y, M)$ is a game where X and Y are infinite, the game has a value if given any $\epsilon > 0$, there is a finite subset of strategies $(x_1, \ldots, x_m)$ such that $M(\xi^{(x)}, y) \geq M(x, y) - \epsilon$ for all $y \in Y$. This is theorem 2.3.3 in Blackwell & Girshick.

We want to prove that given $\epsilon > 0$, $v_I^* - \lambda_I^* \leq 2\epsilon$.
Let

$$S = \{ s(y) = (M(x_1, y), \ldots, M(x_m, y)) \}$$

Assume we have $(x_1, \ldots, x_m) \ni M(\xi^{(x)}, y) \geq M(x, y) - \epsilon$, $\xi^{(x)}$ over $(x_1, \ldots, x_m)$.
Now pick a finite collection of points y as the centers of squares defined by a hatchwork over S:



Each square ~~[illegible]~~ is Δ on a side, $\Delta \leq \epsilon$.

We can make Δ small enough so that when we have n points $(y_1, \ldots, y_n)$ such that there is a j for each y $\ni |S(y) - S(y_j)| \leq \epsilon$. ~~for~~
This implies that given a y, there is a $j \ni$

$$|M(x_i, y) - M(x_i, y_j)| \leq \epsilon \quad \text{for all } i = 1, \ldots, m$$

This is just the i^{th} component of the quantity $|S(y) - S(y_j)|$, so that

$$M(x_i, y) \geq M(x_i, y_j) - \epsilon \quad \text{for all } i = 1, \ldots, m$$

This defines a game $G_0 = (X_0, Y_0, M)$ with good strategies ξ^* and η^* over $X_0 = (x_1, \ldots, x_m)$ and $Y_0 = (y_1, \ldots, y_n)$. The game has a value $v_0 \ni M(\xi, \eta^*) \leq v_0 \leq M(\xi^*, \eta)$. Also, since $M(x, y_j) \leq M(\xi^{(x)}, y_j) + \epsilon$, we have by averaging over η^*:

$$M(x, \eta^*) \leq M(\xi^{(x)}, \eta^*) + \epsilon \leq v_0 + \epsilon$$

Now $\quad \Upsilon_I(\eta^*) = \sup_X M(x, \eta) \leq v_0 + \epsilon$

so $v_r^* \le \Upsilon_r(\eta^*) \le v_0 + \epsilon$.

In the same way we can show that $\lambda_r^* \ge v_0 - \epsilon$ so that

$$\boxed{v_r^* - \lambda_r^* \le 2\epsilon}$$

The procedure of the proof was to pick x's which almost dominated the situation

<u>ϵ-good strategies</u>:

Strategies which guarantee a return within $\epsilon > 0$ of v_r are termed ϵ-good strategies. I.e.,

if $\Lambda_r(\xi) \ge v_r - \epsilon$, then ξ is an ϵ-good strategy
and if $\Upsilon_r(\eta) \le v_r + \epsilon$, " η " " " " .

<u>Theorem</u>: Any game $G = (X, Y, M)$ where X is a closed bounded subset of S_m and Y is a closed bounded subset of S_n, and M is continuous, has a value

<u>Proof</u>:

Let $XY = W$, $w \in W \subset S_{m+n}$; $M(x,y) \leftrightarrow M(w)$.
Given $\epsilon > 0$, $\exists\ \delta > 0 \ni$

$$|M(w_1) - M(w_2)| \le \epsilon \text{ when } |w_1 - w_2| \le \delta$$

This implies

$$|M(x_i, y) - M(x, y)| \le \epsilon \text{ if } |x_i - x| \le \delta$$

The rest of the ~~theorem~~ proof follows as in the previous theorem.

S - games :

Given a game $G = (X, Y, M)$ where $X = (x_1, \ldots, x_m)$ we can define an equivalent game (I_m, S, M) where $I_m = (1, 2, \ldots, m)$ and

$$S = \{ s(y) = (M(x_1, y), \ldots, M(x_m, y)) \} \quad , s \in S \subset S_m$$

$$i \in I_m \quad \text{and} \quad M(i, s) = s_i$$

This can be interpreted geometrically as II choosing a point s in S_m while I chooses a coordinate of S_m. The payoff is the ~~\$~~ magnitude of that coordinate of s.

Mixed extension of an S - game :

$$\Gamma = (Z, R, M)$$
$$Z = \{ \text{all probability distributions } \xi \text{ over } I_m \}$$
$$R = S^* = \text{convex hull of } S$$

$$\xi \in Z \;;\; r \in R \;;\; \text{then} \quad M(\xi, r) = \xi \cdot r = \sum_{i=1}^{m} \xi_i r_i$$

Sample models for statistical decision processes:

(1) Given coins $c_1, \dots, c_N$ and the probability of heads p_i for c_i. Let X_j be a random variable defined as the j^{th} toss of one of these coins (so the value of X_j is H or T).
We want to know which coin we are flipping. When should we stop flipping and make a decision about the coin being tossed; i.e., what is x if we are flipping c_x?

This is an example of a sequential decision process. We are given the X's and the order in which we must observe them. We can stop at any time & make a decision after observing any number of the variables.

(2) Given a set of inputs and outputs possible for transmission, and a set of possible channels through which we transmit

$$s \longrightarrow \boxed{C_j} \longrightarrow r$$

We want to deduce j by observation of the inputs + outputs.

(3) $Z = \{$all possible outcomes$\}$
$\Omega = \{$all possible parameters$\}$

ω determines a probability distribution P_ω over Z.
For each ω,

$$P_\omega(z) \geq 0$$
$$P_\omega(z) = 0 \text{ for all except at most a countable set of } z\text{'s}$$
$$\sum_z P_\omega(z) = 1$$

An event $S \subset Z$, $P_\omega(S) = \sum_{z \in S} P_\omega(z)$

This will be the model for the rest of the course. Ω will be the strategies of nature (ω is the state of nature).
This determines a sample space

$$\mathfrak{Z} = (Z, \Omega, P)$$

<u>Elementary probability properties</u> :

$$0 \le P_\omega(S) \le 1$$
$$P_\omega(S) + P_\omega(T) \ge P_\omega(S \cup T)$$
$$P_\omega(S) + P_\omega(C(S)) = 1 \qquad C(S) = \text{compliment of } S$$

$$P_\omega\left(\bigcup_{i=1}^{\infty}(S_i)\right) = \sum_{i=1}^{\infty} P_\omega(S_i) \qquad \text{if all } S_i \text{ are disjoint.}$$

Often we do not observe z directly but rather $f(z)$. These functions are random variables. Let X be the range space of f which is defined by

$$\mathcal{X} = (X, \Omega, q)$$

$$q = q_\omega(x)$$

Suppose $f(z) = (f_1(z), \dots, f_m(z))$

then $q_\omega(f^*) = q_\omega(f_1^*, \dots, f_m^*) = \{P_\omega(S) : f_i(z) = f_i^*\}$
$\{f_i^*\} = S$

The expectation of a random variable f where f is a vector random variable in m-space is defined if f is absolutely convergent; i.e. if

$$\sum_{z \in Z} |f(z)|\, P_\omega(z) < \infty$$

then

$$E_\omega(f) = \sum_{z \in Z} f(z)\, P_\omega(z)$$

Let φ be defined on X so that we have fns $\varphi(f(z)) \equiv \varphi \circ f$
↑ composition.

then $E_\omega(\varphi \circ f) = \sum_{z \in Z} (\varphi \circ f)(z)\, P_\omega(z)$

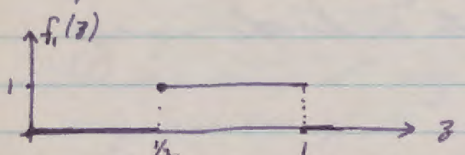
$$= \sum_{f \in F} \varphi(f)\, q_\omega(f).$$

Example of functions partitioning Z:

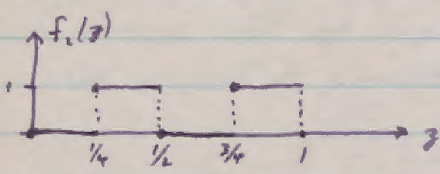
In general, we won't know z itself but some $f(z)$. f partitions Z and the range space of f is $\mathcal{F} = (F, \Omega, \mathcal{F})$. An example of a partition we might encounter is shown below. It is effectively a quantization of Z.

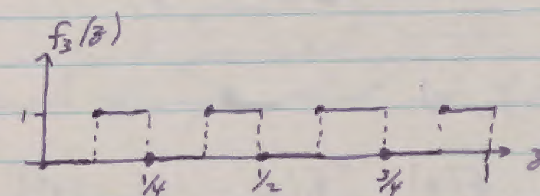
Let $Z = \{ z : 0 \le z \le 1 \}$

Define functions $f_1(z), f_2(z), \ldots \ni$



$$f_1(z) = \begin{cases} 0, & z < 1/2 \\ 1, & z \ge 1/2 \end{cases}$$





If we expand z in the binary expansion

$$z = \sum_{k=1}^{\infty} \frac{a_k}{2^k} \quad , \; a_k = 0, 1 \; , \quad 0 \le z \le 1$$

We call this a *proper* binary expansion if in addition, $a_j = 0$ for all $j > i$ where i can be arbitrarily large.

We see now that $f_k(z) = a_k$, so we can write z as a combination of its binary expansion component:

$$z = (a_1, a_2, \ldots)$$

$$f = (f_1, f_2, \ldots) \in F \quad ; \quad \mathcal{F} = (F, \Omega, \mathcal{F})$$

Now if we cannot pin down z exactly, but only to some quantum division ~~of~~ going up only to f_N; then we are restricted to F^N

$$f^N \in F^N , \quad f^N = (f_1, \ldots, f_N)$$

Single stage experiments:

We are told what set of random variables will be observed. After this observation, the statistician makes his decision; he has no alternative as to what will be observed. We have a sample space

$$\mathfrak{Z} = (Z, \Omega, P)$$

Z has here been narrowed to the appropriate space that we are capable of observing (i.e., F). We also have the action space

$$A = \{\text{all possible decisions}\} = \text{action set}$$

Knowing z, we select an $a \in A$ which we call $d(z) \in A$. Let

$$D = \{d(z) \in A\} = \text{set of decision functions}$$

The sequence of events is: nature chooses $\omega \in \Omega$, z is selected according to $P_\omega \longrightarrow$ we make our decision $d(z) \in A$.

We need some basis for making our decision. In this model we assume a bounded loss function $L(\omega, a)$. We want to minimize $L(\omega, a)$.

For a given decision function $d(z)$, we get a random variable $L(\omega, d(z))$. We now define a *risk function*

$$\rho = \rho(\omega, d) = \sum_{z \in Z} L(\omega, d(z))\, P_\omega(z)$$

We can now regard this situation as a game with nature as player I, the statistician as player II, and the ~~payoff~~ risk function as the payoff which II seeks to minimize.

The game is specified by

$$\mathfrak{Z} = (Z, \Omega, P)\ ;\ A\ ;\ D\ ;\ L$$

which can be reduced to the game triple $\underline{(\Omega, D, \rho)}$.

Example: a communication channel:

$X = \{x_1, \dots, x_b\}$ = set of inputs
$Y = (y_1, \dots, y_b)$ = set of outputs

$q_{x_j}(y_i) = Pr\{y_i \text{ out} / x_i \text{ in}\} = P(y_i | x_i)$

Assume M possible messages, all of length n, $M \leq b^n$
The possible input words are then

$$\left\{\begin{matrix} x_{11}, \dots, x_{1n} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ x_{M1}, \dots, x_{Mn} \end{matrix}\right. \begin{matrix} \to \omega_1 \\ \\ \to \omega_M \end{matrix}$$

Let Z = {all words n letters long}
Ω = {all input messages, indentified by a parameter ω}
$A = \Omega$

The observed output is $z = (y_{z1}, \dots, y_{zn})$

Define $L(\omega, a) = \left\{\begin{matrix} 0; & \omega = a \\ 1; & \omega \neq a \end{matrix}\right\}$

$P_\omega(z) = q_{x_{\omega 1}}(y_{z1}) \cdot q_{x_{\omega 2}}(y_{z2}) \cdot \dots \cdot q_{x_{\omega n}}(y_{zn})$

$\rho(\omega, d) = \sum_{z \in Z} L(\omega, d(z)) \prod_{j=1}^{n} q_{x_{\omega j}}(y_{zj})$

We want to minimize $\rho(\omega, d)$ over d ~~given~~ for all ω.

Conditional expectation:

Given our sample space $\mathcal{Z} = (Z, \Omega, P)$ and an event $S \subset Z$, we want to know the expectation of a function $f = (f_1(z), \ldots, f_N(z))$ defined on Z, given S. Assuming an $\omega \in \Omega$, we restrict our consideration to the case where

$$P_\omega(S) > 0 \quad , \quad P_\omega(S) = \sum_{z \in S} P_\omega(z)$$

The expectation of f given S is defined as

$$\boxed{E_\omega(f|S) = \sum_{z \in S} \frac{f(z)\, P_\omega(z)}{P_\omega(S)}} \qquad \text{if } \sum_{z \in S} |f(z)|\, P_\omega(S) < \infty .$$

Suppose $\exists$ an event $T \subset Z$. Then we can define $f^T(z) = \left\{ \begin{matrix} 1, z \in T \\ 0, z \notin T \end{matrix} \right\}$ so that

$$P_\omega(T|S) = \sum_S \frac{f^T(z)\, P_\omega(z)}{P_\omega(S)}$$

The only values of z contributing to the sum are those common to T and S. Thus we can write

$$\boxed{P_\omega(T|S) = \frac{\sum_{T \cap S} P_\omega(z)}{P_\omega(S)} = \frac{P_\omega(T \cap S)}{P_\omega(S)}}$$

Expectation conditioned on a partition or another random variable:

First we show that conditioned probabilities possess the same properties as unconditioned:

Let our sample space be $\mathcal{Z} = (Z, \Omega, P)$ and let $S \subset Z$. We now construct a new sample space

$$\mathcal{Z}^* = (Z, \Omega_S, P^*) \quad \text{where} \quad P^*_\omega(z) \equiv P_\omega(z|S).$$

Now we investigate the properties of P^*_ω and E^*_ω :

$$\sum_{z \in Z} P_\omega(z|S) = \sum_{z \in S} P_\omega(z|S) + \sum_{z \notin S} P_\omega(z|S) = 1$$

But $\sum_{z \notin S} P_\omega(z|S) = 0$, so $\sum_{z \in S} P_\omega(z|S) = P_\omega(S|S) = 1$

and $\boxed{\sum_{z \in Z} P^*_\omega(z) = \sum_{z \in S} P^*_\omega(z) = 1}$

Similarly, $\boxed{E^*_\omega(f) = E_\omega(f|S)}$, $\omega \in \Omega_s$

Expectation conditioned on a partition:

Let $\mathcal{Z} = (Z, \Omega, P)$ and $\mathcal{S} = (S_1, \dots, S_N)$ be a partition on Z.
Now, the function

$E_\omega(f|S_i)$ is a definite number (constant) for each S_i.

We can now define a function over all Z which takes on the value $E_\omega(f|S_i)$ when $z \in S_i$:

$$\boxed{h(z) \equiv E_\omega(f|\mathcal{S}) \;;\; h(z) = E_\omega(f|S_i) \text{ if } z \in S_i}$$

$h(z)$ takes on values h_i for $z \in S_i$ & $P_\omega(h_i) = P_\omega(S_i)$.

Expectation conditioned on another random variable:

Let $g(\cdot)$ defined on Z generate a partition $\mathcal{S}_g = (S_{g_1}, \dots, S_{g_N})$ on Z such that if $g(z) = g_i$, $z \in S_{g_i}$

Then $E_\omega(f|g_i) = E_\omega(f|S_{g_i})$

or $E_\omega(f|g) = E_\omega(f|g) = E_\omega(f|\mathcal{S}_g)$

Randomized sample spaces:

Let our original sample space be $\mathcal{Z} = (Z, \Omega, P)$ + let us randomize ~~the~~ selection of $\omega \in \Omega$, we get a randomized sample space

$$\mathcal{Z}' = (Z', \mathcal{Z}, q) \qquad z' \in Z', \quad z' = (z, \omega)$$
$$\xi \in \mathcal{Z}, \quad q_\xi(z, \omega) = q_\xi(z')$$

Given an event $S \subset Z'$; $Q_\xi(S) = \sum_{z' \in Z'} q_\xi(z') > 0$,

we define
$$\boxed{E_\xi(f|S) = \sum_{z' \in S} \frac{f(z')\, q_\xi(z')}{Q_\xi(S)}}$$

If we let $S = \{z_0\} \times \Omega = \{(z, \omega) : z = z_0\}$

and $f(z, \omega) = \begin{cases} 1, & \text{if } (z, \omega) = (z_0, \omega_0) \\ 0, & \text{otherwise} \end{cases}$

then
$$E_\xi(f|S) \to \boxed{\xi_z(\omega) \equiv \frac{\xi(\omega)\, P_\omega(z)}{\sum_{\omega \in \Omega} P_\omega(z)\, \xi(\omega)} = \frac{P(z|\omega)\, \xi(\omega)}{\sum_{\omega \in \Omega} P(z|\omega)\, \xi(\omega)}}$$

This is just Bayes theorem.

Or, if we let $f(z, \omega) = L(\omega, d(z))$ then

$$E_\xi(f|S) \to \boxed{T_z(d) = \frac{\sum_{\omega \in \Omega} L(\omega, d(z))\, \xi(\omega)\, P_\omega(z)}{\sum_{\omega \in \Omega} \xi(\omega)\, P_\omega(z)}}$$

Thus, given z and $d(\cdot)$, the quantity $T_z(d)$ is called the a posteriori risk.

$$\boxed{T_z(d) = \sum_{\omega \in \Omega} L(\omega, d(z))\, \xi_z(\omega)}$$

$$\leftrightarrow \quad P_\omega(d) = \sum_{z \in Z} L(\omega, d(z))\, P_\omega(z)$$

Preview of randomized decision-game:

Let $G = (\Omega, D, \rho) \longrightarrow \Gamma = (Z, H, P)$

where η is a randomized decision function $\in H$
d " " non- " " " $\in D$ which uniquely determines all outcomes

If we let Φ be a set of functions on $A \times Z$, $\varphi \in \Phi$, then
$\varphi(a/z) = \varphi_z(a) =$ probability distribution over actions a given z.

$$\rho(\omega, \varphi) = \sum_{a \in A} \sum_{z \in Z} L(\omega, a)\, \varphi_z(a)\, P_\omega(z) \quad ; \quad \varphi_z(a) = \begin{matrix} 1 \\ 0 \end{matrix} \text{ for a non-random decision fn.}$$

Blackwell & Girschick consider only the case where the order of experiments is unimportant. We will consider the more general case where the order is important. The treatment is essentially that of Wald in Statistical Decision Functions.

Statistical Decision Processes:

The first step is to determine the sample space $\mathcal{Z} = (Z, \Omega, P)$
Next we determine a sequence of random variables $f_1(z), \ldots, f_N(z)$ on the sample space. Let

$$X = (f_1(z), \ldots, f_N(z)) = (x_1, \ldots, x_N)$$

and define a new sample space $\mathcal{Z}' = (X, \Omega, q)$ where $x \in X$; $q_\omega(x) = P_n\{x\}$.

We now experiment as follows; Observe a subset of the x_i. We arrange these functions so that x_j is observed only once. If we want to observe the same variable several times, we list each observation as a different variable for notational purposes.

We now try to find a decision rule that will tell us when ~~whether~~ to stop experimenting & take action or what further experiments to observe.

<u>First stage of experimentation</u>:

Let $I_N = (1, \dots, N)$

$$B^e = \{\text{all non-empty subsets of } I_N\}$$

~~We can choose~~ An $\omega \in \Omega$ is picked $\rightarrow P_\omega$ and z is picked according to P_ω. This defines values for $x_1, \dots, x_N$. We can choose to observe any combination of these x's.

Suppose we have observed $x_{i_1}, \dots, x_{i_k}$ in the first stage of the experiment. We now form the set

$$B^e_{i_1, \dots, i_k} = \{\text{all possible sub-sets of } [I_N - (i_1, \dots, i_k)]\}$$

Now define

$$B \equiv A \cup B^e$$

$$B_{i_1, \dots, i_k} \equiv A \cup B^e_{i_1, \dots, i_k}$$

We will do all our selection from B in the first stage, then from $B_{i_1, \dots, i_k}$, etc.

<u>Non-randomized decision function</u>:

Let s_j be a subset of I_N disjoint from all other sub-sets s of I_N and

$$(i_1, \dots, i_k) = \bigcup_{j=1}^{k} s_j$$

s_1 = set of variables observed on first stage of experiment
s_i = " " " " i^{th} stage " "

We now define a decision rule $d(\cdot)$ such that on the k^{th} stage of the experiment

$$d(x; s_1, \dots, s_k) \in B_{i_1, \dots, i_k}$$

which prescribes either an action $a_0 \in A$ or a new set of variables to observe.

Let $d(0)$ be our decision before any experimenting.

(1) If $d(0) = a_0 \in A$, select a_0 without any experimenting.
If $d(0) = s_1 \in B^e$, $s_1 = (j_1, \ldots, j_r)$, then we observe $X_{j_1}, \ldots, X_{j_r}$

(2) If $d(X; s_1) = a_1 \in A$, take action a_1
This is where a fixed experiment stops; it is a special case of sequential testing.
If $d(X; s_1) = s_2 \in B^e_{s_1}$, $s_2 = (l_1, \ldots, l_m)$, then we observe $X_{l_1}, \ldots, X_{l_m}$

(3) Form $d(X; s_1, s_2)$, etc. . . .

At this point, a natural restriction would be
$d(X; s_1, \ldots, s_k) = d(X; s_1', \ldots, s_j')$ if $\bigcup_{i=1}^{k} s_i = \bigcup_{l=1}^{j} s_l'$
but it is not necessary & is occasionally a restriction analytically.

<u>Risk for a non-random decision rule</u>:

Given ω, and that the $z \in Z$ is selected, we want a way to decide what the cost or penalty for future experimentation is.
Let $L(\omega, a)$ be defined if a is the last outcome of the decision rule and is our action, so that $L(\omega, a)$ is the loss.

Define a cost function $c(X; s_1, \ldots, s_k)$
as the cost of an experiment that terminates in k stages in terms of which random variables we observe and their value. $[c(0) = 0]$

The cost and loss functions are often vaguely defined. The decision rule resulting is reasonable if not optimal in spite of this vagueness.

Define the risk for a non-random decision rule as

$$\rho_\omega(d) = \rho_\omega^1(d) + \rho_\omega^2(d)$$

$$\left.\begin{aligned} \rho_\omega^1(d) &= E_\omega(L \mid d) \\ \rho_\omega^2(d) &= E_\omega(c \mid d) \end{aligned}\right\} \text{ for fixed } d, \omega \text{, average over } X \text{ with } f(X).$$

Define

$$h_x(s_1,\dots,s_x, a \mid d) \equiv \begin{cases} 1 \Leftrightarrow d(0)=s_1;\ d(x;s_1)=s_2, \dots, d(x;s_1,\dots,s_k)=a \\ 0 \text{ otherwise} \end{cases}$$

$$h_x(a \mid d) = \begin{cases} 1 \Leftrightarrow d(0)=a \\ 0 \text{ otherwise} \end{cases}$$

Now we can write the risk as:

$$\rho_\omega(d) = \sum_{k=0}^{N} \sum_{s_1,\dots,s_k} \sum_{a\in A} \sum_{x\in X} h_x(s_1,\dots,s_k,a \mid d)\left[L(\omega,a) + C(x;s_1,\dots,s_k)\right] q_\omega(x)$$

<u>Randomized decision functions</u>:

If we let the set of decision functions be D, $d \in D$, we can define experimentation in terms of a game (Ω, D, ρ). Suppose we now mix the strategies $d \in D$ of the statistician. That is we form a probability distribution over "what do we do next" or over ~~...~~ $B = B^e \cup A$.

We define $\varphi(b \mid x; s_1,\dots,s_x)$, $b \in B$, $\varphi \in \Phi$ to be a randomized decision function if & only if

(1) φ is a single valued function for all $b \in B$, all $x \in X$, and any collection of k disjoint subsets $s_1,\dots,s_k \subset I_N$; $k = 1,2,\dots,N$; and $\varphi(b \mid 0) \leftrightarrow k=0$ is independent of x.

(2) $\varphi(\cdot)$ is independent of the values of all x_i if $i \notin \bigcup_{j=1}^{k} s_j$.

(3) For any given x and the disjoint sets $s_1,\dots,s_k$, $\varphi(\cdot \mid x, s_1,\dots,s_k)$ is a probability distribution over B ~~...~~ and so is $\varphi(\cdot \mid 0)$.

(4) If $\bigcup_{j=1}^{k} s_j = (i_1,\dots,i_r) = s$, then $\sum_{b\in B_{i_1,\dots,i_r}} \varphi(b \mid x, s) = 1$

or, ~~...~~ $\varphi(b \mid x, s) = 0$, $b \notin B_{i_1,\dots,i_r}$

Risk function for randomized decision rules:

Let $b^{e(1)} \in B^e$ denote the first stage of experimentation. Then the sequence $b^{e(1)}, \dots, b^{e(\kappa)}, a$ represents a particular sequence of experimentation ~~of~~ which terminated in κ stages with action a.

Given ~~a~~ a sample point x and a decision rule φ, the probability of a specific sequence of experimentation is

$$\mu(b^{e(1)}, \dots, b^{e(\kappa)}, a \mid x, \varphi) = \varphi(b^{e(1)} \mid 0)\, \varphi(b^{e(2)} \mid x;\, b^{e(1)}) \cdots \varphi(a \mid x;\, b^{e(1)}, \dots, b^{e(\kappa)})$$

(A non-randomized decision rule is ~~one~~ a special case of a randomized rule, in which ~~φ~~ $\mu = 1$ for a particular sequence.)

If we now average over x, given the state ω of nature, we get

$$\left[\begin{array}{l} m(b^{e(1)}, \dots, b^{e(\kappa)}, a \mid \omega, \varphi) = \sum_{x \in X} \mu(b^{e(1)}, \dots, b^{e(\kappa)}, a \mid x, \varphi)\, q_\omega(x) \\ \text{and } m(a \mid \omega, \varphi) = \varphi(a \mid 0) \end{array}\right.$$

For a given ω and φ, we can calculate the average loss as

$$\boxed{\rho^1_\omega(\varphi) = \sum_{a \in A} L(\omega, a)\, r(a \mid \omega, \varphi)}$$

where $r(a \mid \omega, \varphi) \equiv r_\omega(a \mid \varphi) \equiv \sum_{\kappa=0}^{N} \sum_{\text{all } b^{e(1)}, \dots, b^{e(\kappa)}} m(b^{e(1)}, \dots, b^{e(\kappa)}, a \mid \omega, \varphi)$.

Similarly, the average cost of experimentation is

$$\boxed{\rho^2_\omega(\varphi) = \sum_{\kappa=0}^{N} \sum_{x \in X} \sum_{b^{e(1)}, \dots, b^{e(\kappa)}, a} \mu(b^{e(1)}, \dots, b^{e(\kappa)}, a \mid x, \varphi)\, q_\omega(x)\, c(x;\, b^{e(1)}, \dots, b^{e(\kappa)})}$$

Then $\boxed{\rho_\omega(\varphi) = \rho^1_\omega(\varphi) + \rho^2_\omega(\varphi)}$

Suggested notation: $\{\text{all}(b^{e(1)}, \dots, b^{e(\kappa)})\} = B^{e(\kappa)}$; $b^e \in B^{e(\kappa)}$
PJW

Example:

Suppose $\Omega = (\omega_1, \omega_2)$ corresponding to two independent binary sequence generators which generat a sequence of N letters & then stop. We observe the output sequence of one and try to decide which generator we are looking at:

$$X = \{\text{all binary words of } n \text{ letters}, \; n \le N\}$$

$$A = (a_1, a_2)$$

$$\boxed{\omega_1} \longrightarrow P_1(x)$$

$$x_i = 0, 1 \quad ; \quad i = 1, \ldots, N.$$

$$\boxed{\omega_2} \longrightarrow P_2(x)$$

$$q_j(x) = q_j(x_1, \ldots, x_N) = \prod_{k=1}^{N} P_j(x_k)$$

Define a *simple* loss function $L(\omega_j, a_k) = \left\{ \begin{array}{l} 0, \; j = k \\ 1, \; \text{otherwise} \end{array} \right\}$

i.e., we are trying to guess ω.

Try a decision rule $d(X; s_1, \ldots, s_K)$ conditioned on ~~[illegible]~~ the

likelihood ratio $\quad l_K = \dfrac{P_1(x_1) P_1(x_2) \cdots P_1(x_K)}{P_2(x_1) P_2(x_2) \cdots P_2(x_K)}$

$$d(0) = 1$$

$$d(x; 1) = \begin{cases} 2, & \beta < l_1 < \alpha \\ a_1, & l_1 \ge \alpha \\ a_2, & l_1 \le \beta \end{cases} \qquad \alpha, \beta \text{ real numbers.}$$

And so on until we find a stage at which l falls outside the region bounded by α and β, or until we reach the last possible stage where

$$d(x; 1, \ldots, N) = \begin{cases} a_1, & l_N \ge 1 \\ a_2, & l_N < 1 \end{cases}$$

i.e., we *must* eventually make a decision even if unhappy about the results.

Utility theory (B&G chap 4):

Under what conditions can we assign a numerical utility function to a set of probability distributions? Let R be some set of outcomes (for example, all (x, s, a) in a statistical game) and let P be the set of all probability distributions over R.

Preference pattern:

A preference pattern is a binary relation between points P_1 and $P_2 \in P$ $\ni$

(1) Given any $P_1, P_2 \in P$, either $P_1 \succeq P_2$ or $P_1 \preceq P_2$ or both, in which case $P_1 \sim P_2$.
$P_1 \succeq P_2 \equiv P_1$ is preferred or indifferent to P_2

(2) The pattern is transitive, i.e., if $P_2 \succ P_1$ and $P_3 \succ P_2$, then $P_3 \succ P_1$

Utility function:

A preference pattern can be described by a utility function $u(\cdot)$ which is a bounded numerical function on P if

(1) $u\left(\sum_m \lambda_m P_m\right) = \sum_m \lambda_m u(P_m)$; $\lambda_m \geq 0$, $\sum_m \lambda_m = 1$

and (2) $u(P_1) \geq u(P_2) \Leftrightarrow P_1 \succeq P_2$

Let $q_r \in P$ and $q_r(r') = \begin{Bmatrix} 1, & r' = r \\ 0, & r' \neq r \end{Bmatrix}$

$$P(r) = \sum_{r' \in R} P(r')\, q_r(r') \quad \text{or} \quad P(\cdot) = \sum_{r' \in R} P(r')\, q_{r'}(\cdot)$$

$$= \sum_{r' \in R} P(r')\, q_{r'}(r)$$

We can now apply condition (1) for the existence of a utility function to get:

$$u(P) = \sum_{r \in R} P(r)\, u(g_r)$$

Define $\quad h(r) \equiv u(g_r)$

so $\quad u(P) = \sum_{r \in R} P(r)\, h(r) = E_P(h)$

<u>Existence conditions for a utility functions</u>:

Theorem 4.2.2 in B+G:

Given a set R, a probability distribution P over R, and a preference pattern $\{\preceq\}$, there exists a utility function if

(1) <u>If</u> $\quad P_{1n} \preceq P_{2n} \quad , \quad n = 1, 2, \ldots$ (representing two distribution from P),

<u>then</u> $\sum_n \lambda_n P_{1n} \preceq \sum_n \lambda_n P_{2n}$ for any sequence of λ's $\ni \lambda_n \geq 0$; $\sum_n \lambda_n = 1$

<u>and if</u> there exists an n_0 such that $\quad P_{1n_0} \prec P_{2n_0} \;, \; \lambda_{n_0} > 0$

<u>then</u> $\quad \sum_n \lambda_n P_{1n} \prec \sum_n \lambda_n P_{2n}$

<u>and</u> (2) If $\quad P_1 \prec P_2 \prec P_3$, then there exist real numbers λ and μ , $0 < \lambda, \mu < 1$ such that

$$\lambda P_1 + (1-\lambda) P_3 \prec P_2 \quad \text{and} \quad \mu P_1 + (1-\mu) P_3 \succ P_2$$

Decision rule selection:

Assume we can represent the experimentation situation as a game
$$G = (\Omega, D, \rho)$$
and its mixed extension
$$\Gamma = (Z, \Phi, \rho)$$

~~Assume~~ Now, which decision rule $d \in D$ do we choose to optimize our return.

Assume
$$\Omega = (\omega_1, \dots, \omega_M)$$
$$D = (d_1, \dots, d_N)$$

Now the quantities $\rho_{\omega_j}(d_k) = a_{jk}$ define a matrix in which the columns correspond to decision rules and rows to ~~[illegible]~~ states of nature. (a_{jk}) is $M \times N$. We want to minimize the payoff found from this matrix.

Baye's rule:

If we know that nature is using a mixed strategy ξ, we choose decision rule d_{j_0} so that

$$\sum_{i=1}^{M} \xi(\omega_i) a_{ij_0} \le \sum_{i=1}^{M} \xi(\omega_i) a_{ij} \quad , \text{all } j = 1, \dots, N$$

That is we minimize the expected payoff to nature by selection of a best decision rule or column of (a_{ij}).

This criterion is nice *if* it is applicable. But we usually don't know what strategy nature is using.

Selection criteria if nature's strategy is unknown:

(1) Laplace method:

Assume all strategies of nature are equally likely and apply Baye's rule, i.e., $\xi(\omega_i) = \frac{1}{m}$. This presents a serious difficulty in deciding what & how many are natures strategies.

(2) Wald method:

Treat nature as an intelligent opponent using a good game strategy to maximize her payoff.

(3) Savage method (minimax regret):

Form a regret matrix (a_{ij}^*) such that

$$a_{ij}^* = a_{ij} - \min_j a_{ij}$$

Here each element a_{ij}^* is our "regret", i.e., what we loose by choosing the rule j instead of the optimal rule for a given strategy i of nature.

We now pick our decision rule ~~by~~ by minimax procedures to minimize our regret.

(4) Hurwitz method:

Given a column j_0 of $(a_{ij}) = (a_{1j_0}, \ldots, a_{mj_0})$, define

$$a_{j_0} \equiv \min_i (a_{1j_0}, \ldots, a_{mj_0})$$

$$A_{j_0} \equiv \max_{i} (a_{1j_0}, \ldots, a_{mj_0})$$

Now define an "optimism parameter" α, $0 \le \alpha \le 1$ to the game. This allows us to assign to each row j a number

$$\lambda_j = \alpha a_j + (1-\alpha) A_j$$

A strategy j_0 is "good" if $\lambda_{j_0} \le \lambda_j$, all $j = 1, \ldots, N$.

\$ <u>Example</u>:

$$(a_{ij}) = \begin{pmatrix} -2 & -1 & 0 & -1 \\ -2 & -1 & -4 & -3 \\ 0 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 \end{pmatrix}$$

(1) <u>Laplace</u>: $\left(-\frac{5}{4} \quad -\frac{4}{4} \quad -\frac{4}{4} \quad -\frac{4}{4}\right) \Rightarrow$ choose d_1

(2) <u>Wald</u> : Saddle point at $a_{22} \Rightarrow$ choose d_2

(3) <u>Regrets</u>:

$$(a_{ij}^{*}) = \begin{pmatrix} 0 & 1 & 2 & 1 \\ 2 & 3 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Saddle point at $a_{34}, a_{44} \Longrightarrow$ choose d_4

(4) <u>Hurwitz</u> : Assume $\alpha > \frac{1}{4}$ + in particular, $\alpha = 7/8$

$$(\lambda_j) = \left(-\frac{7}{4} \quad -1 \quad -\frac{7}{2} \quad -\frac{21}{8}\right)$$

$\max\limits_{j} \lambda_j = \lambda_3 \Rightarrow$ choose d_3

λ_3 dominates for $\alpha > \frac{1}{4}$.

Characteristics desired in a preference pattern:

(1) Complete ordering.

(2) Symmetry: The ordering is invariant under renumbering of rows. $\Leftrightarrow$ no preference as to the state of nature.

(3) Strong domination: If for columns c and c', $c_i < c_i'$ all i, then $c \succ c'$.

(4) Continuity: If for a sequence of matrices $A^{(k)}$, $a_{ij}^{(k)} \to a_{ij}$ for all k as $k \to \infty$; if $c^{(k)} \succ c'^{(k)}$, all k; then $c \succeq c'$.

(5) Linearity: The ordering is unchanged if $a_{ij} \to a_{ij}'$ where $a_{ij}' = \lambda a_{ij} + \mu$, $\lambda > 0$.

(6) Column adjunction: The ordering between old columns is unchanged by the adjunction of a new column to the matrix

(7) Special column adjunction: The ordering between old columns is unchanged by the addition of a new column if no component of the new column is less than the corresponding component of any of the old columns.

(8) Row linearity: Ordering is not changed if a constant is added to any row.

(9) Row duplication: Ordering is not changed if a new row identical to one of the old rows is adjoined to the matrix.

(10) Convexity: If $c = \frac{1}{2}(c' + c'')$ and if $c' \sim c''$ then $c \succeq c' \sim c''$

~~(11)~~

Property	Laplace		Wald		Hurwitz		Savage	
1	X	✓	X	✓	X	✓	X	✓
2	X	✓	X	✓	X	✓	X	✓
3	X	✓	X	✓	X	✓	X	✓
4	X		X	✓	X	✓	X	✓
5	X		X		X	✓	X	
6	X	✓	X	✓	X	✓	O	
8	X	✓	O		O		X	✓
9	O		X	✓	X	✓	X	✓
10	X		X	✓	O		X	✓
7	X		X		X		X	✓

The properties checked under each ~~method~~ rule uniquely specify that ~~method~~ rule.

<u>Theorem, B&G</u>: If $\{\succsim\}$ satisfies 1, 3, 6, ~~7~~ 8, then $\exists\ \xi$ over $\Omega \ni$

$$c \succ c' \Leftrightarrow \sum_j \xi(\omega_j)\, a_{jc} < \sum_j \xi(\omega_j)\, a_{jc'}$$

or all columns are equivalent.

Example $\begin{pmatrix} -2 & -2 \\ -1 & -2 \\ -1 & -\alpha \end{pmatrix}$ $\quad 0 \le \alpha \le 1$

$$\left.\begin{array}{l} \text{If } \alpha = 1, \text{ Wald rule} \rightarrow c_2 \succ c_1 \\ \text{If } \alpha < 1, \quad \ldots \rightarrow c_1 \succ c_2 \end{array}\right\} \text{(4) not satisfied.}$$

Elimination of obviously stupid strategies:

Definitions:

(1) η^* is uniformly better than η if for $\eta, \eta^* \in H$,

$$\rho(\omega, \eta^*) \leq \rho(\omega, \eta) \quad \text{for all } \omega$$

and there exists at least one ω_1 for which $\rho(\omega_1, \eta^*) < \rho(\omega_1, \eta)$

(2) A strategy η is admissable if there is no strategy uniformly better than η.

$A = \{\text{all admissable strategies}\}$

(3) C is a complete class of strategies $[C \subset H]$ if given any $\eta \notin C$, there exists an $\eta^* \in C$ such that η^* is uniformly better than η. All classes A are complete classes.

(4) C_0 is a minimal complete class if C_0 is complete and $C_0 - \{\eta_1\}$ is not complete for $\eta_1 \in C_0$.

Theorem: If C_0 is minimal complete, then $C_0 = A$.

Proof: We show that no $\eta \in C_0$ does not belong to A.

Assume $\eta \in C_0$ but $\eta \notin A$. Then $\exists\, \eta' \ni \eta'$ is uniformly better than η. But $\eta' \notin C_0$ since we must drop it to make C_0 minimal complete. ⊕

Then $\exists\, \eta'' \in C_0$ which is uniformly better than η' which is uniformly better than η.

Thus η'' is uniformly better than η & $\therefore \eta \notin C_0$ which contradicts the assumption.

<u>Bayes class of solutions</u> :

A Bayes class of solutions is complete & imply an a priori distribution (to be proven later).

Let $\Omega = (\omega_1, \dots, \omega_n)$ & put the game in the form of an S-game,
$$\updownarrow$$
$I_n = (1, \dots, n)$, $\omega_j \leftrightarrow j$

Given a $d \in D$, $s = [\rho(\omega_1, d), \rho(\omega_2, d), \dots, \rho(\omega_n, d)]$, $s \in S_m$

The payoff is $M(j, s) = s_j = j^{th}$ component of vector s

$$G = (I_n, S, M) \quad \& \quad \Gamma = (Z, S^*, M)$$

where S^* is the convex hull of S; $\xi \in Z$, $\xi = [\xi(1), \dots, \xi(n)]$

$$M(\xi, s) = \sum_{i=1}^{n} \xi(i)\, s_i = \xi \cdot s$$

<u>Definitions</u> :

(1) A = class of all admissable strategies. If $a \in A \subset S^*$, then $\nexists\ s \in S^* \ni s_i \le a_i$, $i = 1, \dots, n$; <u>and</u> $s_j < a_j$ for at least one $j \in I_n$.

(2) B = Bayes class where $B \subset S^*$.
$b \in B$ if $\exists\ \xi \ni \xi \cdot b \le \xi \cdot s$, $\xi \in Z$, all $s \in S^*$

(3) $Z_\epsilon \subset Z \ni \xi \in Z_\epsilon$ if $\xi(i) \ge \epsilon$, all $i \in I_n$

(4) $Z_+ \subset Z \ni \xi \in Z_+$ if $\xi(i) > 0$, all $i \in I_n$

(5) $D_\epsilon \subset S^*$ where $D_\epsilon = \{$Bayes solutions against $Z_\epsilon\}$

(6) $D \subset S^*$ where $D = \{$Bayes solutions against $Z_+\}$

(7) $\overline{D}$ = closure of D. It includes D and all limits of sequences in D which are outside D.

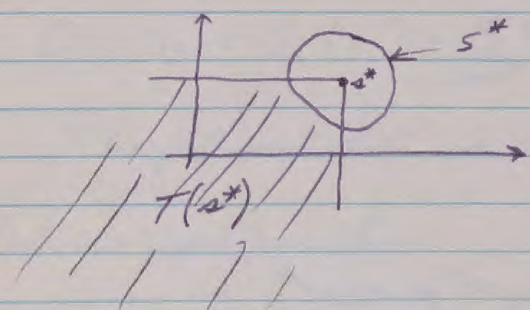
Translation: If $S' = S + r$, $r \in S_n$, the Bayes class $\mathcal{A}$ and its sub-classes are invariant. Other expressions are independent of translation, so the same solution results.

The relative importance of quantities is all that has relevance.

$$D \subset A \subset \overline{D} \subset B$$

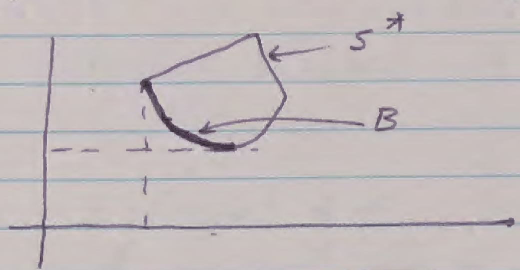
A, $\overline{D}$, B are complete classes & B is closed.

Theorem: $s^* \in B \iff S^* \cap T(s^*) = \emptyset$, $T(s^*) = \{s : s_i \leq s_i^*, \text{all } i\}$, $s^* \in S^*$.



$(\Rightarrow)$ $s \in T(s^*) \Rightarrow s < s^* \Rightarrow \xi \cdot s < \xi \cdot s^* \Rightarrow s$ is uniformly better than s^*. But if $s^* \in B$ then this is false. Hence, $s \notin S^*$ & $S^* \cap T(s^*) = \emptyset$.

$(\Leftarrow)$ $S^* \cap T(s^*) = \emptyset \Rightarrow \xi \cdot s > \xi \cdot s^*$ if $s \in S^*$ & $s^* \notin B$ then if all $s \in S^*$ are not uniformly better than s^*, $s^* \in B$.

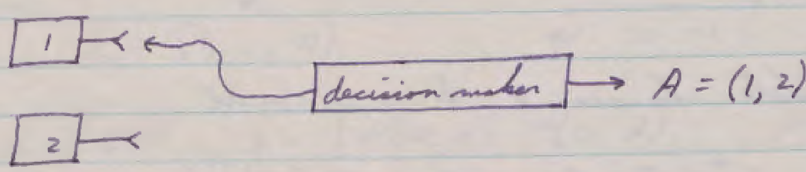


Special case (preview of future of course):

e.g.

Is there a target present or not?

Model:



$\Omega = (1,2)$

Let x_i be a signal received at the i^{th} moment

$X = \{x = (x_1, \dots, x_n)\}$; $P_1(x) = \prod_{i=1}^{n} P_1(x_i)$

$E(n) = \xi(1)\, E_1(n) + \xi(2)\, E_2(n)$; n = length of expt.

$\alpha \equiv Pr\{$guess 2 when 1 is true$\}$

$\beta = Pr\{$guess 1 when 2 is true$\}$

$L(1,1) = L(2,2) = 0$

$L(1,2) > 0$; $L(2,1) > 0$

$E(L) = \xi(1)\, \alpha\, L(1,2) + \xi(2)\, \beta\, L(2,1)$

Define $P(d) = c\, E(n) + E(L)$

We want to minimize $E(n)$ for a given $E(L)$ by choosing d correctly.

Sample space $\mathcal{Z} = (I_2, X, g)$ $\qquad A = (a_1, a_2)$

$x = (x(1), x(2), \dots) \in X$

$x^N = (x(1), \dots, x(N)) \leftrightarrow \{y : y(j) = x(j), j = 1, 2, \dots, N \; ; y \in X\}$

We will toss the coin only a finite number of times, but put no fixed bound on this number.

Let $u \in U$ be a real number & let $f_i(u) \geq 0$, $\sum_T f_i(u) = 1$
$i = 1, 2 \in I_2$

$$Q(x^N | i) = Q_i(x^N) = \sum_{y \in x^N} g_i(y) = \prod_{k=1}^{N} f_i(x(k)) \qquad \text{all } N$$

This just says that we consider each sample point $X(k)$ independent.

Define a cost function: $c(x; s_1, \dots, s_k) = ck$; $c > 0$.

$L(1, a_2) = w_1 > 0$
$L(2, a_1) = w_2 > 0$
$L(2, a_2) = L(1, a_1) = 0$

Assume $\xi = (\xi(1), \xi(2))$

$$\left.\begin{array}{l} \rho(1,d) = 0 \;, \quad \rho(2,d) = w_2 \\ \text{so} \quad \rho(\xi, d) = \xi(2) w_2 \end{array}\right\} \text{ if } d(0) = a_1$$

Assume $E_i(n, d) < \infty$, $i = 1, 2$; $d \in D$
This is really no restriction as it just rules out ∞ experiments.

Define $\left.\begin{aligned}\alpha_1(d) &= \Pr\{a_2 \mid 1,d\}\\ \alpha_2(d) &= \Pr\{a_1 \mid 2,d\}\end{aligned}\right\}$ $\boxed{\rho(i,d) = \alpha_i(d)\,w_i + c\,E_i(n \mid d), \quad i=1,2}$

$$\boxed{\rho(\xi,d) = \xi(1)\rho(1,d) + \xi(2)\rho(2,d)}$$

Our experiment is now defined as a game $G = (I_2, D, \rho)$ or its mixed extension, $\Gamma = (Z, H, \rho)$.

Result: The experiment yields x^n. We compute $Q_2(x^n)$ and $Q_1(x^n)$

$$\xi^n(1) = \frac{Q_1(x^n)\,\xi(1)}{\xi(1)Q_1(x^n) + \xi(2)Q_2(x^n)} = 1 - \xi^n(2) \quad \begin{cases} \le 1 \\ \ge 0 \end{cases}$$

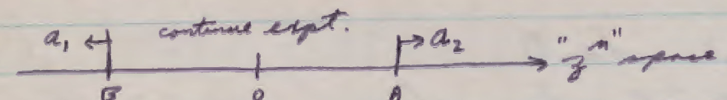
so $$\xi^n(1) = \frac{1}{1 + \frac{\xi(2)}{\xi(1)}\frac{Q_2(x^n)}{Q_1(x^n)}}$$

& $$\frac{Q_2(x^n)}{Q_1(x^n)} = \prod_{k=1}^{n} \frac{f_2(x(k))}{f_1(x(k))}$$

Define $$z^n = \log \frac{Q_2(x^n)}{Q_1(x^n)} = \sum_{k=1}^{n} \log \frac{f_2(x(k))}{f_1(x(k))} \equiv \sum_{k=1}^{n} z(k)$$

Sequential probability ratio test (SPRT)

$d_{AB} \in \{SPRT's\} \subset D$. $\boxed{B < 0 < A}$



We compute z^k as our experiment proceeds. When z^k first goes outside the interval $B \le z^k \le A$, we choose a_1 or a_2

$$d_{AB}(x;n) = \left\{\begin{array}{ll} a_1, & z^n \le B \\ a_2, & z^n \ge A \\ n+1, & B \le z^n \le A \end{array}\right\}$$

Similar to first passage time of random walk problem. z^n is a random variable.

Define : $I_1 = -E_1(z_i) = \sum_u f_1(u) \log \frac{f_1(u)}{f_2(u)}$

$I_2 = E_2(z_i) = \sum_u f_2(u) \log \frac{f_2(u)}{f_1(u)}$

Restrict : $0 < I_1, I_2 < \infty$

As $c \to 0$; $A \to -\log c$; $B \to \log c$

$$E_i(n \mid d_{AB}) \to \frac{-\log c}{I_i}$$

$$\rho(i, d_{AB}) \to \frac{-c \log c}{I_i}$$

$$\alpha_i \to c$$

Theorem: If $P_{iN} = \Pr\{\text{expt termination time } n > N \mid \text{state } i \text{ of nature}\}$

then $\lim_{N\to\infty} P_{iN} = 0$. An SPRT terminates eventually, with probability of unity.

Proof : Let $\gamma = |B| + |A|$

Let $s_1 : z_1, \ldots, z_r$

$\vdots$

$s_k = z_{(k-1)r}, \ldots, z_{kr}$

$$v_k = \sum_{s_k} z_i$$

If the expt never terminates, $|v_k| < \gamma$, all k ; or $v_k^2 < \gamma^2$

Let $P(\gamma, r) = \Pr\{v_k^2 < \gamma^2\}$

$E_i(z_i^2) > 0$; $|E_i(z_i)| < \infty \Rightarrow \sigma_i^2 > 0$; $i = 1, 2$.

But variance $(v_k^2) = r\sigma_i^2$ + we can make this as large as we want. In particular, we can make it so large that for some r, $P(\gamma, r) < 1$ (i.e., v_k must lie outside γ).

Since $P^N(\gamma, r) = P_{iN}$, + $P(\gamma, r) < 1$, $P_{iN} \to 0$ as $N \to \infty$.

Sequential ~~de~~ decision experiments:

$D = \{$ sequential decision functions: $d(0)=1,\ d(x;n) = \begin{Bmatrix} a_1 \\ a_2 \\ n+1 \end{Bmatrix} \}$

$\{SPRT\} \subset D$

For a sample point x and a decision rule d, the course of the experiment is ~~de~~ completely determined. Let $n = n(x,d)$ be the ~~duration~~ time to termination for this sample & decision rule.
$d, x \to n(x,d) \to x^n \to z^n$ where $z^n = \sum_{i=1}^{n} z_i$; $z_i = \log \frac{f_2(x(i))}{f_1(x(i))}$

define $E_i(z^n|d) = E_i(z^{n(x,d)}|d) = \sum_x g_i(x)\, z^{n(x,d)}$

Theorem: If $N = E_2(n|d)$ then

$$\boxed{E_2(z^n|d) = E_2(z^N) = I_2 N} = E_2(z_i)\, E_2(n|d)$$

Proof: Pick N and define $P_{iN} = P_r\{n(x,d) \le N$, given d & state i of nature$\}$

E_{iN} = conditional ~~probability~~ expectation given that $n \le N$

$E_{iN'}$ = " " " " $n > N$

Show for $i=2$:

$P_{2N} E_{2N}[z^n + (z^N - z^n)|d] + (1-P_{2N}) E_{2N'}(z^N|d) = E_2(z^N)$

$\overset{?}{=} I_2 N \overset{?}{=} P_{2N} E_{2N}(z^n|d) + P_{2N} E_{2N}(z^N - z^n|d) + (1-P_{2N}) E_{2N'}(z^N|d)$

$= P_{2N} E_{2N}(z^n|d) + P_{2N}(N - E_2(n|d)) I_2 + (1-P_{2N}) E_{2N'}(z^N|d)$

$\lim_{N\to\infty} P_{2N} E_{2N}(z^n|d) = E_2(z^n|d)$

$\lim_{N\to\infty} P_{2N} E_{2N}(n|d) I_2 = E_2(n|d) I_2$

Let λ_K = probability that test terminates in K trials given $i=2$; d.

$$\infty > E_2(n|d) \geq \sum_{K=N}^{\infty} K\lambda_{2K} \geq N\sum_{K=N}^{\infty} \lambda_{2K} \geq N(1-P_{2N})$$

so $\lim_{N\to\infty} N(1-P_{2N}) \to 0$.

Thus $\boxed{\begin{aligned} I_2 E_2(n|d) &= E_2(z^n|d) \\ (-I_1) E_1(n|d) &= E_1(z^n|d) \end{aligned}}$

So if we know z^n, we know how long the experiment will run.
z^n is an important central quantity.

Defined expectations:

E_i^* = expectation given i st. of nature and $a = a_1$
E_i^{**} = " " " " $a = a_2$

$$\boxed{\begin{aligned} E_1^*(e^{z^n}|d) &= \frac{\alpha_2}{1-\alpha_1} \quad ; \quad E_1^{**}(e^{z^n}|d) = \frac{1-\alpha_2}{\alpha_1} \\ E_2^*(e^{-z^n}|d) &= \frac{1-\alpha_1}{\alpha_2} \quad ; \quad E_2^{**}(e^{-z^n}|d) = \frac{\alpha_1}{1-\alpha_2} \end{aligned}}$$

Jensen's inequality:

$$\boxed{E(u) \leq \log E(e^u)}$$

Proof: $e^{E(u)} \overset{?}{\leq} E(e^u) \to 1 \overset{?}{\leq} E(e^{u'})$, $u' = u - E(u)$; $E(u') = 0$

Now $e^{u'} = 1 + u' + \frac{1}{2}(u')^2 e^{\lambda(u')}$ where $0 \leq \lambda(u') \leq u'$ (Taylor expansion with remainder)

so $E(e^{u'}) = 1 + \frac{1}{2}E[(u')^2 e^{\lambda(u')}] \geq 1$ Q.E.D.

Lower bounds for $E_i(n|d)$:

$$E_1(z^n|d) = (1-\alpha_1)\, E_1^*(z^n|d) + \alpha_1\, E_1^{**}(z^n|d)$$

$$E_1(z^n|d) \le (1-\alpha_1)\log\frac{\alpha_2}{1-\alpha_1} + \alpha_1 \log\frac{1-\alpha_2}{\alpha_1}$$ by Jensen's inequality.

& $$E_2(z^n|d) \ge (1-\alpha_2)\log\frac{\alpha_1}{1-\alpha_2} + \alpha_2 \log\frac{1-\alpha_1}{\alpha_2}$$

so

$$\boxed{\begin{aligned} I_1 E_1(n|d) &\ge (1-\alpha_1)\log\frac{1-\alpha_1}{\alpha_2} + \alpha_1 \log\frac{\alpha_1}{1-\alpha_2} \ge 0 \\ I_2 E_2(n|d) &\ge (1-\alpha_2)\log\frac{1-\alpha_2}{\alpha_1} + \alpha_2 \log\frac{\alpha_2}{1-\alpha_1} \ge 0 \end{aligned}}$$

Optimality of the SPRT:

> If d is a test belonging to D with strengths α_1, α_2 and
> d_s is an SPRT " " " " " " "
> then
> $$E_i(n|d_s) \le E_i(n|d) \quad , i=1,2 \quad , \text{all } \alpha_1, \alpha_2.$$

Proof for case when n is large (or A, B are large):

Let $u' = z^n - E_1^*(z^n|d)$

Then $E_1^*(e^{u'}|d) = 1 + \frac{1}{2}E_1^*[(u')^2 e^{\lambda(u')}] \le 1 + \frac{1}{2}E_1^*[(u')^2]$

since $u' < 0$ so $\lambda(u') < 0$.

$$1 \le E_1^*(e^{u'}|d) \le 1 + \tfrac{1}{2}E_1^*[(u')^2]$$

so

$$E_1^*(z^n|d) \ge \log E_1^*(e^{z^n}|d)$$

or $$E_1^*(z^n|d) \ge \log E_1^*(e^{z^n}|d)\left[1 + \frac{1}{2}\frac{E_1[(u')^2]}{E_1^*(e^{-z^n}|d)}\right]$$

Asymptotically, for large n or large A, B,

$$\boxed{E_1^*(z^n|d) \to \log E_1^*(e^{z^n}|d)}$$

Second solution of SPRT :

This approach is generally applicable but not generally useful.

Consider a finite set $z_1, \ldots, z_M$

Approximate each z_i by $\boxed{z_i = k_i \lambda}$. We can always make λ small enough to make this reasonable.

Define $P_{ij} = P_r\{z' = j\lambda \mid \text{nature } i\}$, $j = 0, \pm 1, \pm 2, \ldots$, $z' \in (z_1, \ldots, z_M)$

If $A, B \neq k\lambda$, then d_{AB} is equivalent to $d_{A'B'}$; $A' = [A]^*$, $-B' = [-B]^*$

Let $A = a\lambda$, $a > 0$
$B = b\lambda$, $b < 0$

$$\xrightarrow[\;-3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3\;]{} \; y^m = \frac{z^m}{\lambda}$$

Define:

$\alpha_{iy} = P_r\{\text{error} \mid \text{start at } y;\ \text{nature } i\}$

$\alpha_{i0} = \alpha_i$

$\beta_{iy} = P_r\{\text{terminating in one step with wrong step} \mid \text{nature } i\}$

$\beta_{1y} = P_{1,a-y} + P_{1,a-y+1} + \cdots + P_{1,a-1}$

$\beta_{2y} = P_{2,b-y} + P_{2,b-y-1} + \cdots + P_{2,b+1}$

$$\alpha_{iy} = \underbrace{\sum_{k=b+1}^{a-1} \alpha_{ik} P_{i,k-y}}_{\text{prob don't terminate but end at } k} + \underbrace{\beta_{iy}}_{\text{one step termination.}} \qquad b < y < a$$

This gives a set of non-homogeneous, ~~non-singular~~ equations. The determinant is non-singular, so a solution exists.

Suppose we now define

$$\alpha_{1y} = \begin{Bmatrix} 1, & y \geq a \\ 0, & y \leq b \end{Bmatrix} \quad ; \quad \alpha_{2y} = \begin{Bmatrix} 1, & y \leq b \\ 0, & y \geq a \end{Bmatrix}$$

Then $\alpha_{iy} = \sum_{k=-\infty}^{\infty} \alpha_{ik} P_{i,k-y}$ — homogeneous equations

Reduces to previous non-homogeneous equations.

Assume $-\nu\lambda \le z' \le \mu\lambda \rightarrow \nu+\mu$ equations.

Solution: Assume $\alpha_{iy} = C s^{y}$, $s = s(i)$

$$C s^{y} = \sum_{k=-\infty}^{\infty} C s^{k} P_{i,k-y}$$

$$1 = \sum_{-\infty}^{\infty} s^{k-y} P_{i,k-y} = \sum_{-\infty}^{\infty} s^{j} P_{ij} = \sum_{-\nu}^{\mu} s^{j} P_{ij}$$

This gives a $(\nu+\mu)$-order polynomial. Solving this we get $\mu+\nu$ roots assuming all are different. s_{ik}, $k = 1, \dots, \mu+\nu$.

$$\boxed{\alpha_{iy} = \sum_{k=1}^{\mu+\nu} C_{ik} S_{ik}^{y}}$$

C_{ik} are $\mu+\nu$ constants to allow satisfaction of boundary conditions.

If $s_{i1} = s_{i2}$, sol $\rightarrow C_1 s_{i1}^{y} + C_2 y s_{i1}^{y}$

$$\alpha_i = \alpha_{i0} = \sum_{k=1}^{\mu+\nu} C_{ik}$$

Define $h_{iy}(n) = P_r\{\text{expt terminates in exactly } n \text{ steps} \mid \text{nature } i, \text{ start at } y\}$

$$\gamma_{1y} = P_{1,b-y} + P_{1,b-y-1} + \cdots$$

$$\gamma_{2y} = P_{2,a-y} + P_{2,a-y+1} + \cdots$$

$$h_{iy}(1) = \beta_{iy} + \gamma_{iy}$$

$$h_{iy}(n+1) = \sum_{k=b+1}^{a-1} h_{ik}(n) P_{i,k-y}$$

$$h_{ik}(0) = 1, \quad k \ge a, \; k \le b$$

$$h_{ik}(n) = 0, \quad k \ge a, \; k \le b, \; n \ge 1$$

Limits for $c \to 0$

$$\frac{\alpha_1}{\dfrac{\xi(2)\ c}{I_2 w_1 \xi(1)}} \to 1 \quad ; \quad \frac{\alpha_2}{\dfrac{\xi(1)\ c}{I_1 w_2 \xi(2)}} \to 1$$

$$\frac{E(n|d)}{-\dfrac{\log c}{I}} \to 1$$

$$E(L) = \xi(1)\alpha_1 w_1 + \xi(2)\alpha_2 w_2 \approx c\left[\frac{\xi(1)}{I_1} + \frac{\xi(2)}{I_2}\right]$$

$$\boxed{\text{Define } \frac{1}{I} = \frac{\xi(1)}{I_1} + \frac{\xi(2)}{I_2}}$$

$$\frac{\log E^{-1}(L)}{I\, E(n|d)} \to 1 \quad ; \quad E(L) \to K e^{-I\, E(n|d)}$$

since $\log E(L) = \log c - \log I \Rightarrow E(L) \to c \Rightarrow$

This is analogous to the probability of error approaching zero in a communications channel as the code length grows.

Define a fixed length test : $d_{fn}(x^j) = \begin{Bmatrix} j+1,\ j < n \\ a_1 \\ a_2 \end{Bmatrix}$, c, w_1, w_2

$c \to 0 \Rightarrow \dfrac{\log E_f^{-1}(L)}{I^f\, \cancel{E}\, n \cancel{\xi}} \to 1$, $I_f > 0$

so $E_{fn} \approx K e^{-I_{\cancel{f}}^{f} n}$

Channel n-chotomy:

N channels; inputs $x(\kappa) \in (1, \ldots, M)$; outputs $y(\kappa) \in (1, \ldots, M_0)$; argument κ indicates κ^{th} input of expt.

$P_\kappa^{(x(l))}(y(l))$ = ~~the~~ probability distribution associated with the κ^{th} channel for the ~~~~ input $x(l)$ defined over the set of outputs $y(l)$.

$$P_\kappa^{(x(l))}(y(l)) = P_\kappa^{(j)}(i) \quad \text{all } \kappa. \qquad \begin{array}{l} i = 1, \ldots, M_0 \\ j = 1, \ldots, M \\ \kappa = 1, \ldots, N \end{array}$$

Three functions can completely specify a decision rule $d \leftrightarrow (\nu, g, a)$. For the κ^{th} stage of the expt ($\kappa \geq 1$),

$$\nu = \nu(x, y, \kappa) = \nu'(x^\kappa, y^\kappa) = \begin{cases} 0, & \text{continue} \\ 1, & \text{stop} \end{cases}$$
$$\nu(x, y, 0) = \nu'(0)$$

$$g(x, y, \kappa) = g'(x^\kappa, y^\kappa) = \left\{ \begin{array}{c} 1 \\ \vdots \\ M \end{array} \right\} \text{ try this input if } \nu(\kappa) = 1$$
$$g(x, y, 0) = g'(0)$$

$$a(x, y, \kappa) = a'(x^\kappa, y^\kappa) = \left\{ \begin{array}{c} a_1 \\ \vdots \\ a_N \end{array} \right\} \text{ select this decision if } \nu(\kappa) = 0$$
$$a(x, y, 0) = a'(0)$$

Define $L(j, a_\kappa)$ = loss when a_κ is result and true nature is j.

$E_i(L|d)$ = expected loss given nature i, rule d.

Consider the product space XY. Define S_κ as a set

$$\boxed{\begin{aligned} S_\kappa &= \{(x, y) \mid \nu(x, y, \kappa) = 1 \,;\, \nu(x, y, j) = 0 \text{ if } j < \kappa\} \\ &= \{\text{all infinite sequences of observations of input \& corresponding output for which the expt. terminates on the } \kappa^{th} \text{ step.}\end{aligned}}$$

Successive events are independent, so we can write

$$\boxed{P_i(x, y, K) = P_i^{(x(1))}(y(1))\, P_i^{(x(2))}(y(2)) \cdots P_i^{(x(K))}(y(K)) = P_r\{y^K \mid x^K, \text{nature } i\}}$$

$$E_i(L|d) = \sum_{K=0}^{\infty} \sum_{S_K} L(i, a(x,y,K))\, P_i(x,y,K)$$

$$\rho(\xi, d) = \sum_{i=1}^{N} \xi(i)\left[E_i(L|d) + c\, E_i(n|d)\right]$$

$$E_i(n|d) = \sum_{K=0}^{\infty} K \sum_{S_K} P_i(x,y,K)$$

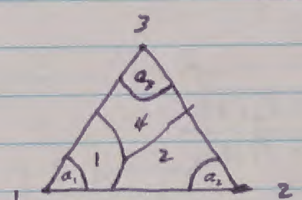
d determines S_K and $a(x,y,K)$ & through these, affects $\rho(\xi,d)$ via $E_i(n|d)$ and $E_i(L|d)$.

If we fix L, and maximize $\frac{E(L|d)}{E(n|d)}$ over possible rules d, this is a Bayes rule for some c.

If $E(L|D) = V$, $E(n|d) = f(V)$, then $\rho = V + c\, f(V)$. This describes a trading curve between cost and accuracy.

~~The problem is The~~

The problem can be displayed geometrically as in the dichotomy. E.g., for 3 dimensions



Elimination of x in notation:

We will consider rules such that $\nu'(0) = 0$, $g'(0) = x(1)$.

$$\left.\begin{array}{l} d \to x(1) \\ d, y(1) \to x(2) \\ d, y(1), y(2) \to x(3) \\ \quad \cdots \end{array}\right\}$$

If we know d and all $y(i)$, $i = 1, \ldots, k$ [i.e., y^k], then we know X^k also. That is, there exists a function γ such that

$$\gamma(y, m, d) = x^m$$

So we can now write

$$\boxed{d \leftrightarrow \nu(y,k), \; g(y,k), \; a(y,k)}$$

and $\boxed{P_i(y^k) = P_i^{(x(1))}(y(1)) \cdots P_i^{(x(k))}(y(k))}$ for given d where $x(1) = g(0)$
$x(m+1) = g(y,m)$

$$E_i(L|d) = \sum_{k=0}^{\infty} \sum_{S_k} L(i, a(y,k)) P_i(y^k)$$

$$\begin{aligned} S_k &\equiv \{y \mid \nu(0) = 0; \; \nu(y,1) = \cdots = \nu(y,k-1) = 0; \; \nu(y,k) = 1\} \\ &= \{y \mid \text{expt terminates on } k^{th} \text{ trial}\} \end{aligned}$$

A posteriori distributions:

d generates y^n from which we compute ξ_n which is the a posteriori distribution over ~~each~~ the channels after n expt. trials. [$\xi_0 = \xi$ = a priori dist]

$$\xi_k = \frac{\xi(i) P_i(y^k)}{\sum_{j=1}^{N} \xi(j) P_j(y^k)} = \frac{1}{1 + \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{\xi(j) P_j(y^k)}{\xi(i) P_i(y^k)}}$$

Conditions imposed on n-chotomy

(1) $P_i^{(j)}(y') = 0 \Leftrightarrow P_k^{(j)}(y') = 0$, all k

This guarantees that we will have to make tests

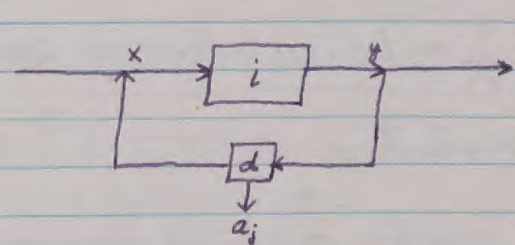
(2) $\nexists\, i, j, k \ni P_i^{(j)}(y') = P_k^{(j)}(y')$, $i \neq k$, all y'.

This assures that *any* test will eventually reduce $E(L)$ to zero as a limit.

(3) $L(i, a_j) = 1 - \delta_{ij}$ [Dirac delta]

So we can write $E(L|d) = P_E(d)$ = probability of error.

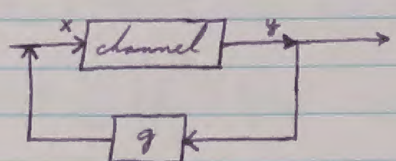
Test:



We just put the decision box tester around the channel; it generates inputs & eventually pops out the decision.

Channel processes:

A channel process is characterized by a channel and a "go-ahead rule" g connected around the channel:



E.g., any finite state Markov process is a channel process where $g(y, k) = x(k+1) = y(k)$.

Let $Y^n = \{y^n\} = \{$all output sequences of length $n\}$

$Y' = \{y'\} = \{$all ~~[illegible]~~ outputs$\}$

$$c_j(y(i)) = \begin{cases} 1, & x(i) = j \\ 0, & \text{otherwise} \end{cases}$$

$$m_j = \sum_{i=1}^{m} c_j(y(i)) = \#\text{ of times input } j \text{ is used in } x^m.$$

$f_1(y'), \ldots, f_M(y')$ are functions with numerical values which are defined on Y'.

Theorem: Given any $\epsilon > 0$, $\exists\ \delta > 0 \ni$ for all m,

$$\boxed{Pr\left\{\left|\sum_{i=1}^{m}\sum_{j=1}^{M} c_j(y(i))\, f_j(y(i)) - \sum_{j=1}^{M} m_j E_j(f_j)\right| > m\epsilon\right\} \le 2e^{-\delta m}}$$

where the quantity in magnitude brackets is

$$\left[f_{x(1)}(y(1)) + f_{x(2)}(y(2)) + \cdots + f_{x(m)}(y(m))\right] - \left[\cancel{m_{x(1)} E_{x(1)}(f_{x(1)}) + \cdots + m_{x(m)} E_{x(m)}(f_{x(m)})}\ m_1 E_1(f_1) + m_2 E_2(f_2) + \cdots + m_M E_M(f_M)\right].$$

Or, if we let $\varphi_1 = f_{x(1)}(y(1))$, $\quad \overline{\varphi^1_*} = \sum_{y'} \varphi^1_* P(y')$, $\quad \varphi' = f_1$

$$[\varphi_1 + \varphi_2 + \cdots + \varphi_m] - [m_1 \overline{\varphi^1_*} + m_2 \overline{\varphi^2_*} + \cdots + m_M \overline{\varphi^M_*}]$$

Where $\boxed{E_j(f_j) = \sum_{Y'} P^{(j)}(y')\, f_j(y')}$

Let $h(y(i)) \equiv \sum_{j=1}^{M} c_j(y(i))\left[f_j(y(i)) - E_j(f_j)\right]$

and ~~where~~ $F_m \equiv \sum_{i=1}^{m} h(y(i))$

then the theorem becomes

$$Pr\{|F_m| > m\epsilon\} \le 2e^{-\delta n}$$

Proof:

Lemma: Let $v \in (v_1, \dots, v_N)$; $\bar{v} = 0$; $P(v_i) = P_i$

Let $G(s) = \sum_{i=1}^{N} P(v_i) e^{s v_i}$, s real

Let $a = \max_i |v_i|$

Now the lemma says: $G(s) \le G^*(s) \equiv \frac{1}{2}(e^{as} + e^{-as})$

Proof of lemma:

Let $v_i = q_i a + (1-q_i)(-a) = (2q_i - 1)a$, $0 \le q_i \le 1$

$$e^{s v_i} = e^{s q_i a + s(1-q_i)(-a)} \le q_i e^{as} + (1-q_i) e^{-as}$$ since e^x is convex.

so $G(s) \le \sum_i P_i q_i e^{as} + P_i (1-q_i) e^{-as}$

$$q_i = \frac{v_i}{a} + (1-q_i)$$

$$\left.\begin{array}{l} \sum_i P_i q_i = \sum_i P_i \left[\frac{v_i}{a} + (1-q_i)\right] = \sum P_i (1-q_i) \\ \sum_i P_i q_i + \sum_i P_i (1-q_i) = 1 \end{array}\right\} \sum P_i q_i = \sum P_i (1-q_i) = \frac{1}{2}$$

so $\boxed{G(s) \le \frac{1}{2}(e^{as} + e^{-as}) \equiv G^*(s)}$

Now, $E[h(y(i))] = \sum_{Y'} h(y(i)) P(y(i) | y^{i-1}) = \sum_{Y'} h(y(i)) P^{x(i)}(y(i))$

$$h(y(i)) = \sum_{Y'} c_j(y(i)) [f_j(y(i)) - E_j(f_j)] = \sum_{Y_i} c_j(y(i)) \left[f_j(y(i)) - \sum_{Y'} f_j(y(i)) P^{(j)}(y(i))\right]$$

$$= f_{x(i)}(y(i)) - E_{x(i)}(f^{P}_{x(i)})$$

so $E(h(y(i)) = \sum_{Y'} \left[f_{x(i)}(y(i)) - E_{x(i)}(f_{x(i)})\right] P^{(x(i))}(y(i))$

$$= E_{x(i)}(f_{x(i)}) - E_{x(i)}(f_{x(i)}) \underbrace{\sum_{Y'} P^{(x(i))}(y(i))}_{=1} = 0.$$

Hence, $\boxed{E[h(y(i))] = 0}$

Now let $G(s)$ be the generating function

$$G(s) = E(e^{sF_m}) = \sum_{Y^m} P(y^m) e^{sF_m}$$

$$P(y^m) = P(y(1)) P(y(2)|y^1) P(y(3)|y^2) \dots P(y(m)|y^{m-1})$$

$$e^{sF_m} = \prod_{i=1}^{m} e^{s h(y(i))}$$

or, $$G(s) = \sum_{Y^1} e^{s h(y(1))} P(y(1)) \sum_{Y^1} e^{s h(y(2))} P(y(2)|y^1) \dots \sum_{Y^1} e^{s h(y(m))} P(y(m)|y^{m-1})$$

$$G(s) \le [G^*(s)]^m$$

Let $\ln G(s) = \gamma(s)$, $\ln G^*(s) = \gamma^*(s)$

then $G(s) = E(e^{sF_m}) \le [G^*(s)]^m = e^{m\gamma^*(s)}$

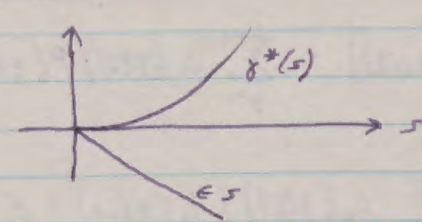
$$G(s) = E(e^{sF_m}) \ge P(U) E(e^{sF_m}|U), \quad U \subset Y^m$$

$$U = \{y^m | F_m \ge m\epsilon\} \subset Y^m$$

so $e^{\gamma(s)} \ge P(U) e^{sm\epsilon}$

or $P(U) \le e^{\gamma(s) - sm\epsilon} \le e^{m[\gamma^*(s) - \epsilon s]}$

Now $$\left.\frac{d\gamma^*(s)}{ds}\right|_0 = \left.\frac{\frac{dG^*(s)}{ds}}{G^*(s)}\right|_0 = 0$$



Therefore, there is an $s = s_0 \ni \gamma^*(s_0) - \epsilon s_0 = -\delta_0 < 0$

& $P(U) \le e^{-m\delta_0}$

If $U' \equiv \{y^m | -F_m \ge m\epsilon\}$,

$P(U') \le e^{-m\delta_1}$

Now let $\delta = \min(\delta_0, \delta_1)$, and we have the result of the theorem:

$$\boxed{P(U) + P(U') = P_r\{|F_m| \geq m\epsilon\} \leq 2e^{-m\delta} \quad , \quad \delta > 0}$$

<u>Definition</u>: Now define the functions $f_j(y(i))$ to be a particular function:

$$\boxed{f_j(y(i)) \equiv -\log P^{(j)}(y(i))}$$

Then

$$H^{(j)} = -\sum_{Y'} P^{(j)}(y(i)) \log P^{(j)}(y(i))$$

$$n(y) \equiv \text{time for expt to stop given the output } y$$

$$h_k \equiv \sum_{j=1}^{M} h_{jk}$$

The result of the previous theorem can now be expressed in two additional forms:

$$(1) \quad P_r\left\{\left|\frac{\log P^{-1}(y^m)}{m} - \frac{1}{m}\sum_{j=1}^{m} H^{(j)} m_j\right| > \epsilon\right\} \leq 2e^{-m\delta}$$

or

$$(2) \quad P_r\left\{\left|\frac{1}{m}\sum_{k=1}^{m} h_k\right| \geq \epsilon\right\} \leq 2e^{-m\delta}$$

Now, for a particular channel where N is a particular value of n,

$$G_n(y) = G_n = \sum_{i=1}^{n(y)} \sum_{j=1}^{M} c_j(y(i)) f_j(y(i)) = \sum_{i=1}^{n(y)} \sum_{j=1}^{M} \left[h_{ji} + c_j(y(i))\, E_i(f_j) \right]$$

$$E(G_n) = E\left\{ \sum_{j=1}^{M} \left(\sum_{i=1}^{n(y)} h_{ji} \right) \right\} + \sum_{j=1}^{M} E(n_j) E_j(f_j)$$

where $E(n_j)$ is the average number of inputs (j)

$$E(G_n) = \sum_{j=1}^{M} E(F_{nj}) + \sum_{j=1}^{M} E(n_j) E_j(F_j) \quad , \quad \boxed{F_{nj} \equiv \sum_{i=1}^{n(y)} h_{ji}}$$

Theorem : $E(G_n) = \sum_{j=1}^{M} E(n_j) E_j(F_j)$

To prove this, we show that $E(F_{nj}) = 0$ via $E(F_{Nj}) = 0$.

Lemma : $E(F_{Nj}) = E(F_{Nj} \mid n \le N) P(n \le N) + E(F_{Nj} \mid n > N) P(n > N) = 0$

$|E(F_{Nj} \mid n > N) P(n > N)| \le A\, N\, P(n > N)$ where $\boxed{A \equiv \max h_{ji}}$

$E(n) < \infty \Rightarrow \lim_{N \to \infty} N P(n > N) = 0$

$\therefore \quad \lim_{N \to \infty} |E(F_{Nj} \mid n > N) P(n > N)| = 0$

or $\lim_{N \to \infty} \left[E(F_{Nj}) - E(F_{Nj} \mid n \le N) P(n \le N) \right] = 0$ so $\boxed{\lim_{N \to \infty} E(F_{Nj} \mid n \le N) P(n \le N) = 0}$

Now suppose we hang a "stop rule" ν on the output of the process, so that the expt. stops at n. We can continue on after this & any y will be acceptable beyond this point.

Let $F'_{nj} \equiv F_N - F_n$

Then if $n \le N$,

$$F'_{nj} = \sum_{i=n+1}^{N} h_{ji}$$

$$E(F_{Nj} \mid n \le N) = E(F_{nj} \mid n \le N) + E(F'_{nj} \mid n \le N)$$

But $E(F'_{nj} \mid n \le N) = 0$ since $E(h_{ji}) = 0$ for all i.

So $\lim_{N \to \infty} E(F_{Nj} \mid n \le N) = \lim_{N \to \infty} (F_{nj} \mid n \le N) = 0$ by previous lemma

$\therefore\ E(F_{nj}) = 0$.

Continuation of channel N-chotomy (low cost limit):

N channels : distribution over channels $\xi = (\xi(1), \dots, \xi(N))$

$A = (a_1, \dots, a_N)$; ~~guess~~ choose $a_i \leftrightarrow$ guess channel i is present.

$$L(j, a_k) = \begin{cases} 1, & k \neq j \\ 0, & k = j \end{cases}$$

Given test d, this determines $E(L|d) = P_e(d)$

and $\rho(\xi, d) = P_e(d) + c\,E(n|d)$

Now let everything be fixed but c in our rules; d_c is the rule for cost c. Or, we may just write d, understanding that c is the only variable.

Results to be shown:

(1) $\displaystyle \lim_{c \to 0} \frac{\log P_e(d)}{E(n|d)} = I$ $\Big\}$ $\underline{P_e \to K e^{-I E(n)}}$

(2) $\displaystyle \lim_{c \to 0} E(n|d) = \infty$

(3) If $\xi_n(1) \to 1$, then for some $i \neq 1$, $\xi_n(i) >> \xi_n(j)$, $j \neq i$, $j \neq 1$.

i.e., we can rule out all ~~—~~ possible errors except one with very high probability.

$$\boxed{\text{Standard inequality: } \quad \sum P_1 \log P_1 \geq \sum P_2 \log P_1 \quad \text{if } P_2 \neq P_1 \text{ for all arguments}}$$

Definitions

$$H_{ik}^{(j)} \equiv -\sum_{y'} P_i^{(j)}(y') \log P_k^{(j)}(y') \equiv \text{cross entropy per step for input } (j)$$

$q \equiv (q_1, \ldots, q_M) \equiv$ probability distribution over the input symbols

$I_{ik}^{(j)} \equiv H_{ik}^{(j)} - H_{ii}^{(j)}$; the first subscript present corresponds to the channel present.

Now, using the distribution q, define

$$I_i \equiv \max_q \underbrace{\min_{k \neq i} \sum_{j=1}^{M} I_{ik}^{(j)} q_j}_{\text{minimum rate of growth of } \xi^{\prime\prime}}$$

$\frac{1}{I} \equiv \sum_{i=1}^{N} \frac{\xi(i)}{I_i}$; I is a measure of how easy it is to make a good guess. It is the maximum average information we can gain per stage of experimentation

Restrictions:

(1) $P_i^{(j)}(y') = 0$ if & only if $P_k^{(j)}(y') = 0$, all k

$\Rightarrow H_{ik}^{(j)} < \infty$; $I_{ik}^{(j)} > 0$

(2) There is no input (j) for which

$P_i^{(j)}(y') = P_k^{(j)}(y')$ for all y'

so $0 < I_i < \infty$

Theorem: Given any $\epsilon > 0$, there exists $A_\epsilon > 0$ such that if $A_\epsilon < E(n|d) < \infty$, then

$$\frac{\log P_e^{-1}(d)}{E(n|d)} \leq (1+\epsilon) I$$

This theorem says that if $E(n|d)$ is bounded below, then we have also an upper bound for $\frac{\log P_e^{-1}(d)}{E(n|d)}$

We now set $P_e(d_c) = P_e(d)$ where d_c is a Bayes rule & want to show that $E(n|d_c) \leq E(n|d)$. This shows that a Bayes rule gives the maximum of $\frac{\log P_e^{-1}(d)}{E(n|d)}$; Then if the theorem is true for d_c, it is true for all d.

Proof: (Assume d_c in all expressions where it is needed but not specified)

$$E_i\left[\frac{P_k(y^n)}{P_i(y^n)}\middle| Q_i\right] = \sum_{y \in Q_i} \frac{\frac{P_k(y^n)}{P_i(y^n)} P_i(y^n)}{P_i(Q_i)} = \frac{P_k(Q_i)}{P_i(Q_i)}$$

$\boxed{Q_i \equiv \{y \mid \text{decision } a_i \text{ is reached}\}}$; $\overline{Q_i} \cup Q_i = \{y\}$

$$E_i\left[\frac{P_k(y^n)}{P_i(y^n)}\middle| \overline{Q_i}\right] = \frac{P_k(\overline{Q_i})}{P_i(\overline{Q_i})}$$

$$E_i[\log P_k^{-1}(y^n)] = \sum_{j=1}^{M} E_i(n_j) H_{ik}^{(j)}$$

Now $E(\log u) \leq \log E(u)$,

$$\text{so } E_i\left[\log \frac{P_k(y^n)}{P_i(y^n)}\right] = E_i\left[\log \frac{P_k(y^n)}{P_i(y^n)}\middle| Q_i\right] P_i(Q_i) + E_i\left[\log \frac{P_k(y^n)}{P_i(y^n)}\middle| \overline{Q_i}\right] P_i(\overline{Q_i})$$

$$\leq P_i(Q_i) \log \frac{P_k(Q_i)}{P_i(Q_i)} + P_i(\overline{Q_i}) \log \frac{P_k(\overline{Q_i})}{P_i(\overline{Q_i})}$$

$$\leq P_i(Q_i) \log P_k(Q_i) + \log 2$$

Let $P_k(e) = P_{ek} = \sum_{i \neq k} P_k(Q_i) \geq P_k(Q_i)$, $i \neq k$

$$\underline{P_{ei} = 1 - P_i(Q_i)}$$

Now, for any given $\epsilon' > 0$, there exists $B_{\epsilon'}$ such that

$P_{ek} \leq \frac{\epsilon'}{1+\epsilon'}$ for all k if $E(n) \geq B_{\epsilon'}$

so $E_i\left[\log \frac{P_k(y^n)}{P_i(y^n)}\right] \leq \frac{1}{1+\epsilon'} \log P_{ek} + \log 2$, $k \neq i$ and $E(n) \geq B_{\epsilon'}$.

since $P_i(Q_i) \geq \frac{1}{1+\epsilon'}$.

Now, the left hand side can be written

$$E[\ \ ] = -\sum_{j=1}^{M} I_{ik}^{(j)} E_i(n_j) = \frac{1}{1+\epsilon''} \log(2 P_{ek})$$

or

$$2 P_{ek} \geq \exp\left\{-(1+\epsilon'') \sum_{j=1}^{M} E_i(n_j) I_{ik}^{(j)}\right\} , \quad i \neq k$$

so $2(N-1) P_{ek} \geq \sum_{i \neq k} \exp\{\ \ \}$

$$\Rightarrow P_{ek} \geq \frac{1}{2N} \sum_{i \neq k} \exp\{\ \}$$

$$\Rightarrow \sum_{k=1}^{N} \xi(k) P_{ek} = P_e \geq \frac{1}{2N} \sum_{k=1}^{N} \xi(k) \sum_{i \neq k} \exp\{\ \ \}$$

$$P_e \geq \frac{1}{2N} \sum_{k=1}^{N} \xi(k) \sum_{i \neq k} \exp\left\{-(1+\epsilon'') E_i(n) \sum_{j=1}^{M} \frac{E_i(n_j)}{E_i(n)} I_{ik}^{(j)}\right\}$$

Recalling that $E_i(n) = \sum_{j=1}^{M} E_i(n_j)$,

let $\xi^* = \min_k \xi(k) \neq 0$

Then
$$P_e \geq \frac{\xi^*}{2N} \sum_{i=1}^{N} \exp\{-(1+\epsilon'') I_i E_i(n)$$

Now minimize $E_i(n)$ ~~and~~ over i and select this particular $\underline{i}$:

Minimization is by Lagrange multipliers subject to the constraint
$$\sum_k \xi(k) E_k(n) = E(n) \geq B_{\epsilon'}.$$

The result of this minimization is
$$P_e \geq \frac{\xi^*}{2N} e^{-(1+\epsilon'')E(n) I}$$

or $$\log P_e^{-1} \leq (1+\epsilon'')E(n) I + \log\left(\frac{2N}{\xi^*}\right)$$

or $$\frac{\log P_e^{-1}}{E(n)} \leq I\left[\underbrace{1+\epsilon'' + \frac{\log \frac{2N}{\xi^*}}{I B_{\epsilon''}}}_{\epsilon}\right]$$

Now, if $\epsilon'' \to 0$, $\frac{1}{B_{\epsilon}} \to 0$ & $\therefore$ $\epsilon \to 0$, so

$$\frac{\log P_e^{-1}}{E(n)} \leq I(1+\epsilon) \quad ; \quad E(n) \geq A_\epsilon$$

Definition: $e_d \equiv \dfrac{\log P_e^{-1}(d)}{I\,E(n|d)}$ = efficiency of rule d.

Theorem: Given any $\epsilon > 0$, there exist decision rules arbitrarily long, such that

$$\boxed{\frac{\log P_e^{-1}(d)}{E(n)} \geq I(1-\epsilon)}$$

Method of experimentation: Pick Δ $(0<\Delta<1)$. If $\xi_m(i) \geq 1-\Delta$, pick d_i. Look at the a posteriori distribution after each r stages; i.e., at ξ_r, ξ_{2r}, ...

Always assume that the channel that is present is the one we are currently closest to. Pick the input symbol that gives the fastest mean drift rate toward absorption. This is how we determine $x(1)$, ... [want $\max_i \min_j$ drift]

Go-ahead rule:

Recall: $I_{ix}^{(j)} = H_{ix}^{(j)} - H_{ii}^{(j)} = \sum_{y'} P_i^{(j)}(y') \log \dfrac{P_i^{(j)}(y')}{P_x^{(j)}(y')}$

$$I_i = \max_{q_j} \min_{x \neq i} \sum_{j=1}^{M} q_j I_{ix}^{(j)}$$

Let $S_i = \{ \text{integers } x = x(i) \mid I_i = \sum_{j=1}^{M} q_{ij} I_{ix(i)}^{(j)}$

Assume there exists a r large enough so that $q_{ij} = \dfrac{r_{ij}}{r}$, all $\begin{matrix} i = 1, \dots, N \\ j = 1, \dots, M \end{matrix}$

$$\sum_j r_{ij} = r$$

Now, the go-ahead rule says: at any time lr, $(l = 0, 1, \dots)$,

let $\xi_{lr}(i) = \max_k \xi_{lr}(k)$ (i.e., we are closest to channel (i)). and experiment with

r_{i1} inputs (1), r_{i2} inputs (2), ..., r_{iM} inputs (M) for the next time period.

Theorem: Given any $\epsilon > 0$, there exists decision rules arbitrarily long, such that

$$\nu(y^m) = 0, \quad m \neq lr$$
$$\nu(y^{lr}) = \begin{cases} 1, & \max_i \xi_{lr}(i) \geq 1-\Delta \\ 0, & \text{otherwise} \end{cases}$$

Then $P_i\left\{ \left| \frac{1}{m} \log P_k^{-1}(y^m) - \sum_{j=1}^{M} \frac{m_j}{m} H_{ik}^{(j)} \right| > \epsilon \right\} \leq 2e^{-\delta m}$ $\quad$ *

~~#~~ implies: (1) $E(n) < \infty$

(2) Given $\epsilon > 0$, there exists $m >$ some m_0 such that

$$P_i\left\{ \left| \frac{m_j}{m} - \frac{r_{ij}}{r} \right| > \epsilon \right\} \leq A e^{-\delta m}$$

(3) Given $\epsilon > 0$, $\Delta \leq \Delta_\epsilon$, then

$$\frac{E_i(n_j)}{E_i(n)} \geq (1-\epsilon) \frac{r_{ij}}{r}$$

Proof of (1):

Define $S_{im} \equiv$ {all output sequences satisfying * }, $i = 1, \ldots, N$

$$S_{im} = \bigcap_{k=1}^{N} \left\{ y^m \,\middle|\, \left| \frac{\log P_k^{-1}(y^m)}{m} - \sum_{j=1}^{M} \frac{m_j}{m} H_{ik}^{(j)} \right| \leq \epsilon \right\}$$

$$P_i(\overline{S_{im}}) \leq 2Ne^{-\delta m}$$

If $y^m \in S_{im}$, then $\frac{\log P_k^{-1}(y^m)}{m} \leq \sum_{j=1}^{M} \frac{m_j}{m} H_{ik}^{(j)} + \epsilon$

or $\log P_k^{-1}(y^m) \geq \sum_{j=1}^{M} m_j \left[H_{ik}^{(j)} - \epsilon \right]$

or $P_k(y^m) \leq \exp\left\{ -\sum_{j=1}^{M} m_j \left[H_{ik}^{(j)} - \epsilon \right] \right\}$

and $\dfrac{1}{P_i(y^m)} \le \exp\left\{ \sum_{j=1}^{M} m_j \left[H_i^{(j)} + \epsilon \right] \right\}$

then $\boxed{\begin{array}{l} \dfrac{P_k(y^m)}{P_i(y^m)} \le \exp\left\{ - \sum_{j=1}^{M} m_j \left[I_{ik}^{(j)} - 2\epsilon \right] \right\} \\ \text{If } i = k \text{, we can make this close to unity} \\ \quad\; i \ne k \text{, it} \to 0 \text{ for large } m \text{, since } I_{ik}^{(j)} - 2\epsilon > 0 \end{array}}$

Now recall that $\xi_m(i) = \dfrac{1}{1 + \sum_{k \ne i} \frac{\xi(k)}{\xi(i)} \frac{P_k(y^m)}{P_i(y^m)}}$

So we can now choose m_0' such that if $m > m_0'$ and if $y^m \in S_{im}$, then

$$\xi_m(i) > 1 - \Delta$$

If $m \ge m_0' + r$, then $y^m \in \overline{S_{i,m-r}}$ or $y^m \notin S_{i,m-r}$

and $P_i(n = m) \le P_i(\overline{S_{i,m-r}}) \le 2N e^{-\delta(m-r)}$

$$E_i(n) = \sum_{m=0}^{\infty} m P_i(n=m) = \underbrace{\sum_{m=0}^{m_0'+r-1} [\ ]}_{\uparrow \text{ finite}} + \sum_{m_0'+r}^{\infty} [\ ]$$

so $E_i(n) \le \sum_{0}^{m_0'+r-1} [\ ] + \sum_{m_0'+r}^{\infty} m\, 2N e^{-\delta(m-r)} < \infty$.

Hence, $E_i(n) < \infty$

Proof of (2): Given any $\epsilon > 0$, there exists some m_0 such that if $m > m_0$, then

$$P_i\left\{ \left| \frac{m_i}{m} - \frac{r_{ii}}{r} \right| > \epsilon \right\} \le A e^{-m\delta}.$$

Define m_0 such that if $y \in S_{im}$, and $m > m_0$, then $\xi_m(i) > 3/4$

" m_0' " " " " " . . . $m > m_0'$, then $\xi_m(i) > 1 - \Delta$

$$P_i\{ \xi_m(i) < 1 - \delta \} \le 2N e^{-\delta m}, \quad m > m_0'$$

Define $T_{im} = \{ y \mid \max_k \xi_m(k) = \xi_m(i) \}$

If $m \ge m_0$, $\xi_m(i) > 3/4$, $y \in S_{im} \subset T_{im}$

$\Rightarrow \overline{T_{im}} \subset \overline{S_{im}}$

$\Rightarrow P_i(\overline{T_{im}}) \le P_i(\overline{S_{im}}) \le 2N e^{-\delta m}$

~~Define $v_{ik} = \overline{T_{i,k-1}} \cap$~~

Define $v_{ik} \equiv \overline{T_{i,k-1}} \cap T_{ik} \cap T_{i,k+1} \cap \cdots$

If the sequence belongs to v_{ik} from k on, we are causing the correct sequence

$P_i(v_{ik}) \le P_i(\overline{T_{i,k-1}}) \le 2N e^{-\delta(k-1)}$ if $k > m_0 + 1$

$v_{ik} \cap v_{ij} = \phi$, $k \ne j$; $\sum_{k=0}^{\infty} P_i(v_{ik}) = 1$

Now let $m = cr$, c is an integer

Assume $y \in v_{ik}$ for $k \le m$. We are using r_{ij} beyond k, so

$$m_j \le \left(\frac{m-k+r}{r}\right) r_{ij} + k$$

$$(m-k-r)\frac{r_{ij}}{r} \le m_j \quad \text{or} \quad \frac{r_{ij}}{r} - \frac{k+r}{m} \le \frac{m_j}{m}$$

$$\text{so} \quad \frac{r_{ij}}{r} - \frac{k+r}{m} \le \frac{m_j}{m} \le \frac{r_{ij}}{r} + \frac{\frac{-k\,r_{ij}}{r} + k + r_{ij}}{m} \le \frac{r_{ij}}{r} + \frac{k+r}{m}$$

$$\text{or} \quad y \in v_{ik} \Rightarrow \left|\frac{m_j}{m} - \frac{r_{ij}}{r}\right| \le \frac{k+r}{m}$$

Let $[x]$ = largest integer less than $=x$, so $[m\epsilon] \le m\epsilon$

If $k+r \le [m\epsilon]$, or $k \le [m\epsilon] - r$, then

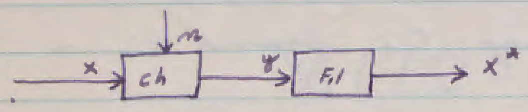
$$\left|\frac{m_j}{m} - \frac{r_{ij}}{r}\right| \le \epsilon \quad \text{and also,} \quad [m\epsilon] - r \ge m.$$

$$P_i\left\{\left|\frac{m_j}{m} - \frac{r_{ij}}{r}\right| > \epsilon\right\} = \sum_{k=0}^{\infty} P_i\left\{\left|\frac{m_j}{m} - \frac{r_{ij}}{r}\right| > \epsilon,\ v_{ik}\right\} \quad \text{(a joint prob)}$$

$$= \sum_{k=[m\epsilon]-r+1}^{\infty} P_i\{\ \} \le \sum_{k=[m\epsilon]-r+1}^{\infty} P_i(v_{ik}) \le \sum_{k=[m\epsilon]-r+1}^{\infty} 2N e^{-\delta(k-1)}$$

$$\le \frac{2Ne^{\delta r} e^{-[m\epsilon]}}{1-e^{-\delta}} \le \frac{2Ne^{\delta(r+1)}}{1-e^{-\delta}} e^{-m\epsilon}$$

Prediction and filtering:



$x = (x(1), x(2), \ldots)$ is a stochastic sequence which is the input to the channel

n = noise

We observe $y(1), y(2), \ldots, y(N)$, and want to predict $x(N+T)$.
I.e., we want x^* to be a prediction of $x(N+T)$.

From the statistics, we get the distributions

$$P[y(1), \ldots, y(N), x(N+T)]$$
$$P[x(N+T)]$$
$$P[x(N+T) / y(1), \ldots, y(N)]$$

Define a loss function $L(x, x^*)$

Now, $$\sum_{x(N+T)} P[x(N+T) / y(1), \ldots, y(N)]\ L(x, x^*) = \inf_{x^{**}} \sum_{x(N+T)} P[\ \ ]\ L(x, x^{**}).$$

Thus we have the result that $x^* = F(y^N)$ and we build our filter to realize this function.

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