SAMPLED AND QUANTIZED
SYSTEMS
6.54

C.T. WHITEHEAD SPRING '61

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Name

6.54 : SAMPLED AND QUANTIZED SYSTEMS

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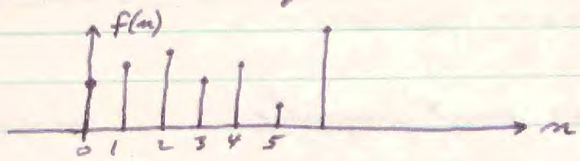
$\frac{1}{4}$ quizzes $\frac{1}{4}$ homework
$\frac{1}{2}$ final

A <u>discrete system</u> is a system whose inputs and outputs are <u>discrete time functions</u>. A discrete time function is a function which takes on some value at the discrete time intervals $n = 0, 1, 2, \ldots$ These times need have no relation to real time, but may be based on successive occurrences of some external event (clock).

<u>Representation</u>:

We can represent the discrete time functions in two ways.

(1) $f(n) \longrightarrow \boxed{} \longrightarrow g(n)$



$f(n)$ is the (number) value which the input time function f takes on at time n.

(2) Another representation is to consider the function f as a "vector" $\underline{f}$ whose elements are the values of the input at all times

$\underline{f} \longrightarrow \boxed{} \longrightarrow \underline{g}$

$$\underline{f} = \{\text{all numbers } f(n) \text{ for } n = 0, 1, 2, \ldots\}$$
$$= [f(0), f(1), f(2), \ldots]$$

Here the system can be regarded as a transformation on the vector $\underline{f}$ into the vector $\underline{g}$.

$\underline{f}^k = \underline{f}$ with all values shifted to the right (occurring at a later time) by k time units.

$$f(n) \leftrightarrow \underline{f}$$
$$f(n-k) \leftrightarrow \underline{f}^k$$

Linearity:

We will restrict our study to linear systems. That is, if $\underline{f_1} \rightarrow \underline{g_1}$, and $\underline{f_2} \rightarrow \underline{g_2}$, then

$$a\underline{f_1} + b\underline{f_2} \longrightarrow a\underline{g_1} + b\underline{g_2}$$

and $a f_1(n) + b f_2(n) \rightarrow a g_1(n) + b g_2(n)$

Time invariance:

We will also study only time-invariant systems; i.e., systems whose characteristics do not change with time.

$$\text{If } \underline{f} \rightarrow \underline{g} \text{ , then } \underline{f}^{K} \rightarrow \underline{g}^{K}$$

or, if $f(n) \rightarrow g(n)$, then $f(n-K) \rightarrow g(n-K)$

Unit ~~vector~~ or unit impulse:

Let $\delta(n)$ be a discrete time function which has non-zero value only at the origin where $\delta(0) = 1$

$$\delta(n) = \begin{Bmatrix} 1, & n = 0 \\ 0, & \text{otherwise} \end{Bmatrix}$$

Impulse response:

If we put the unit impulse into our system input, the response of the system is called the ~~input response~~ impulse response.

$$\underline{\delta} \rightarrow \boxed{} \longrightarrow \underline{h}$$

Convolution summation:

Given the impulse response $\underline{h}$ of the system, we can find the response of the system to any arbitrary input:

First note that for any particular value of κ, $f(\kappa)$ is a constant, so $f(\kappa)\underline{\delta}^{\kappa} \to f(\kappa)\underline{h}^{\kappa}$ since $\underline{\delta}^{\kappa} \to \underline{h}^{\kappa}$. Further, we can sum over all values of κ to get

$$\left.\begin{aligned} &\sum_{\kappa} f(\kappa)\,\underline{\delta}^{\kappa} = \underline{f} \\ \text{and}\quad &\sum_{\kappa} f(\kappa)\,\underline{\delta}^{\kappa} \longrightarrow \sum_{\kappa} f(\kappa)\,\underline{h}^{\kappa} \end{aligned}\right\} \text{so}\quad \underline{f} \to \underline{g} = \sum_{\kappa} f(\kappa)\,\underline{h}^{\kappa} = \underline{f} \underset{\text{convolution}}{*} \underline{h}$$

Or, as an explicit formula for $g(n)$,

$$\boxed{g(n) = \sum_{\kappa} f(\kappa)\,h(n-\kappa) \quad : \text{convolution summation}}$$

see below

Characterization by impulse response:

We have just seen that by virtue of linearity & time-invariance, we can find the response of the system to any arbitrary discrete time function by knowing only the impulse response of the system. Thus we can completely ~~do~~ characterize the system by its impulse response. This will be done schematically as

$$\underline{f} \to \boxed{\underline{h}} \to \underline{g} \qquad \text{or} \qquad \underline{f} \circ\!\xrightarrow{\;h\;}\!\circ\, \underline{g}$$

where $\underline{g} = \underline{f} * \underline{h}$

Realizability:

We will restrict ourselves to realizable systems (i.e., those whose impulse response is zero for $n<0$). As a convenience, we will further assume that our input time functions begin at $n=0$ (i.e., they are zero before $n=0$).

Under these assumptions the above convolution summation becomes:

$$\boxed{g(n) = \sum_{\kappa=0}^{n} f(\kappa)\,h(n-\kappa)}$$

Properties of convolution

(1) $\underline{f} * \underline{h} = \underline{h} * \underline{f}$

$$\underline{f} * \underline{h} = \sum_{k=0}^{n} f(k)h(n-k) = \sum_{j=n}^{0} f(n-j)h(j) = \sum_{j=0}^{n} h(j)f(n-j) = \underline{h} * \underline{f}$$

(2) $\underline{f} * [\underline{h} * \underline{g}] = [\underline{f} * \underline{h}] * \underline{g} \equiv \underline{f} * \underline{h} * \underline{g}$

$$\cancel{\underline{h} * \underline{g} = \sum_{k=0}^{n} h(k)g(n-k) \quad \underline{f} * [\underline{h} * \underline{g}] = \sum_{j=0}^{n} f(j) \sum_{k=0}^{n} h(k)g(n-k)}$$

$$\cancel{\underline{f} * [\underline{h} * \underline{g}] = \sum_{j=0}^{n} f(j) \sum_{k=0}^{n} g(k)h(n-k) \quad \text{since} \quad \underline{h} * \underline{g} = \underline{g} * \underline{h}}$$

$$[f * h] * g = \sum_{j} \sum_{k} f(k)\, h(j-k)\, g(n-j)$$

$$= \sum_{m} \sum_{k} f(k)\, h(m)\, g(n-k-m) \qquad \text{if } m = j-k$$

$$= \sum_{k} f(k) \sum_{m} h(m)\, g(n-k-m)$$

$$= f * [h * g]$$

(3) $\underline{f} * [\underline{h} + \underline{g}] = \underline{f} * \underline{h} + \underline{f} * \underline{g}$

$$\underline{f} * [\underline{h} + \underline{g}] = \sum_{k} f(k)\left[h(n-k) + g(n-k)\right] = \sum_{k} f(k)h(n-k) + \sum_{k} f(k)g(n-k).$$

Elementary combinations of systems:

(1) Cascaded systems

$$\underline{f} \circ \xrightarrow{\underline{h}_1} \underset{\underline{e}}{\circ} \xrightarrow{\underline{h}_2} \circ\, \underline{g} \quad \equiv \quad \underline{f} \circ \xrightarrow{\underline{h}_1 * \underline{h}_2} \circ$$

$$\underline{e} = \underline{f} * \underline{h}_1 \quad \& \quad \underline{g} = \underline{e} * \underline{h}_2 = \underline{f} * [\underline{h}_1 * \underline{h}_2]$$

Hence, the overall impulse response of two cascaded systems is the convolution of the responses of the two systems

(2)
$$\begin{tikzcd}[row sep=small] \underline{f}_1 \circ \arrow[dr, "\underline{h}_1"] & \\ & \circ\, \underline{g} \\ \circ \arrow[ur, "\underline{h}_2"'] & \end{tikzcd} \qquad \underline{g} = \underline{f}_1 * \underline{h}_1 + \underline{f}_2 * \underline{h}_2$$

When the outputs of two systems converge at one node, we mean that the value at that node is the sum of the outputs of the two systems.

(3) Parallel systems

$$\begin{tikzcd} \underline{f} \circ \arrow[r, bend left, "\underline{h}_1"] \arrow[r, bend right, "\underline{h}_2"'] & \circ\, \underline{g} \end{tikzcd} \quad \equiv \quad \underline{f} \circ \xrightarrow{\underline{h}_1 + \underline{h}_2} \circ\, \underline{g}$$

$$\underline{g} = \underline{f} * \underline{h}_1 + \underline{f} * \underline{h}_2 = \underline{f} * [\underline{h}_1 + \underline{h}_2]$$

(4) Feedback loop

$$\begin{tikzcd} \underline{f} \circ \arrow[r, "\underline{\delta}"] & \underset{\underline{e}}{\circ} \arrow[r, "\underline{\delta}"] \arrow[loop below, "\underline{h}"] & \circ\, \underline{g} \end{tikzcd}$$

$$\underline{g} = \underline{e} = \underline{f} + \underline{g} * \underline{h}$$

$$\underline{f} = \underline{g} * [\underline{\delta} - \underline{h}]$$

Now to solve for $\underline{g}$, we are stuck as we have not defined an inverse operation for convolution.

We are now motivated to somehow transform our "vectors" to scalers in hopes that ~~this~~ with these transforms we can find a simple inverse to convolution.

Z-transforms:

Define the z-transform of f as

$$\boxed{\mathcal{Z}(f) \equiv \sum_{k=0}^{\infty} f(k)\, z^{k} \equiv F(z)}$$

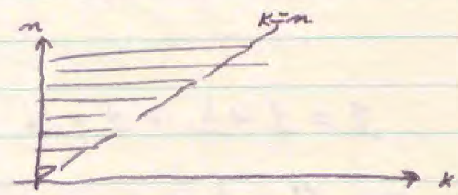
This transformation on f gives a scaler quantity which preserves the individual components of f. That is, the components of f can be found from $F(z)$ by a Taylor series expansion of $F(z)$ about $z = 0$.

System transfer function:

$$f \xrightarrow{\;h\;} g \qquad\qquad g(n) = \sum_{k=0}^{n} f(k)\, h(n-k)$$

$$\mathcal{Z}\{g\} = \sum_{n=0}^{\infty} z^{n} g(n) = \sum_{n=0}^{\infty} z^{n} \sum_{k=0}^{n} f(k)\, h(n-k) \equiv G(z)$$

$$G(z) = \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} f(k)\, z^{k} z^{n-k}\, h(n-k)$$



$$G(z) = \sum_{k=0}^{\infty} f(k)\, z^{k} \sum_{j=0}^{\infty} h(j)\, z^{j} \quad , \quad j \equiv n-k$$

$$\boxed{G(z) = H(z)\, F(z)}$$

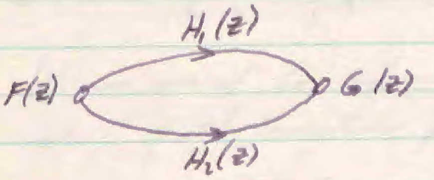
Thus, we have found a new way of characterizing a system. We call $H(z)$ the <u>system transfer function</u>. The output transform is just the product of the input transform and the system transfer function:

$$F(z) \xrightarrow{\;H(z)\;} G(z) = F(z)\, H(z)$$

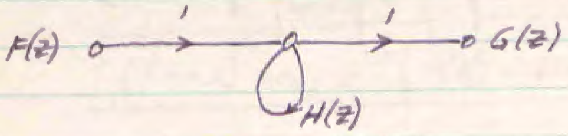
Properties of transforms:

(1) Our elementary system combinations first discussed on page 5 can now be ~~extended~~ described in the transform domain:

$$F(z) \circ\!\xrightarrow{H_1(z)}\!\circ\!\xrightarrow{H_2(z)}\!\circ\, G(z) \quad \equiv \quad F(z) \circ\!\xrightarrow{H_1(z)H_2(z)}\!\circ\, G(z)$$



$$\equiv \quad F(z) \circ\!\xrightarrow{H_1(z)+H_2(z)}\!\circ\, G(z)$$



$$G(z) = F(z) + G(z)H(z)$$

$$G(z)[1 - H(z)] = F(z)$$

$$G(z) = F(z) \; \frac{1}{1-H(z)}$$

$$\equiv$$

$$F(z) \circ\!\xrightarrow{\frac{1}{1-H(z)}}\!\circ\, G(z)$$

Here, our transformation has enabled us to solve the feedback situation by transforming convolution in the time domain to multiplication in the transform domain. We can perform the inverse of multiplication and hence can solve for $G(z)$ directly.

Elementary transform pairs: $f(n) \leftrightarrow F(z)$

(1) If $f(n) = \delta(n)$, $F(z) = \sum_{n=0}^{\infty} \delta(n) z^n = 1$

$\boxed{\delta(n) \leftrightarrow 1}$ Identity branch

If $f(n) = \delta(n-k)$, $F(z) = \sum_{n=0}^{\infty} \delta(n-k) z^n = z^k$

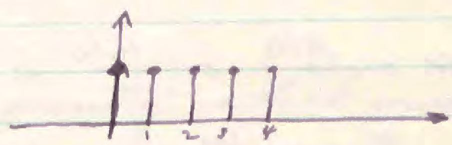
$\boxed{\delta(n-k) \leftrightarrow z^k}$ delay of k time units

(2) $f(n) = a^n$, $F(z) = \sum_{n=0}^{\infty} a^n z^n = \sum_{n=0}^{\infty} (az)^n = \frac{1}{1-az}$, $|az| < 1$

$\boxed{a^n \leftrightarrow \frac{1}{1-az}}$ <u>increasing or decreasing geometric series</u>

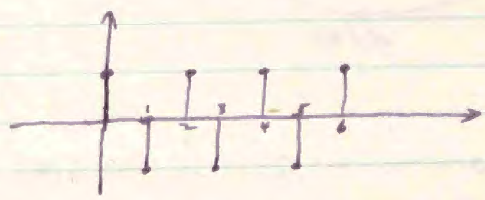
In particular, if $a = 1$,

$\boxed{f(n) = 1 \leftrightarrow \frac{1}{1-z}}$ <u>unit step</u>



If $a = -1$, $f(n) = (-1)^n$

$\boxed{(-1)^n \leftrightarrow \frac{1}{1+z}}$



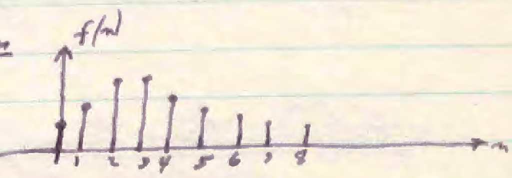
(3) $F(z) = \sum_{n=0}^{\infty} f(n) z^n$

$\frac{dF(z)}{dz} = \sum_{n=0}^{\infty} f(n)\, n\, z^{n-1}$

$z \frac{dF(z)}{dz} = \sum_{n=0}^{\infty} n f(n) z^n \Longrightarrow \boxed{n f(n) \leftrightarrow z \frac{dF(z)}{dz}}$

(4) $f(n) = n a^n$, then from (2) & (3) above, $F(z) = z \frac{d}{dz}\left[\frac{1}{1-az}\right]$

$\boxed{n a^n \leftrightarrow \frac{z}{(1-az)^2}}$ <u>te^{-t} type function</u>



In particular, if $a = 1$, $f(n) = n$

$\boxed{n \leftrightarrow \frac{z}{(1-z)^2}}$ <u>unit ramp</u>

(5) Obviously, $a f_1(n) + b f_2(n) \leftrightarrow a F_1(z) + b F_2(z)$

(6) Shifts in the time domain

$$f(n-k) \leftrightarrow \sum_{n=0}^{\infty} f(n-k) z^n = \sum_{j=-k}^{\infty} f(j) z^{j+k} = z^k F(z) \quad \text{if } k \geq 0$$

$\boxed{f(n-k) \leftrightarrow z^k F(z)}$ <u>delay of k time units</u> $(k \geq 0)$

If we want to <u>advance</u> the function in time, retaining only that part of the function for which $n \geq 0$,

$$f(n+k) \leftrightarrow \sum_{n=0}^{\infty} f(n+k) z^n = z^{-k} \sum_{j=k}^{\infty} f(j) z^j = z^{-k}\left[F(z) - f(0) - z f(1) \cdots - z^{k-1} f(k-1)\right]$$

$\boxed{f(n+k) \leftrightarrow z^{-k}\left[F(z) - f(0) - z f(1) \cdots - z^{k-1} f(k-1)\right] \quad k \geq 0}$ <u>time advance of k</u>

<u>Example</u>: $f(n) = \left(\frac{1}{2}\right)^n, \; n \geq 0$
$h(n) = \left(\frac{1}{3}\right)^n, \; n \geq 0$

$$g(n) = \sum_{k=0}^{n} f(k) h(n-k) = \sum_{k=0}^{n} \left(\frac{1}{2}\right)^k \left(\frac{1}{3}\right)^{n-k} = \left(\frac{1}{3}\right)^n \sum_{k=0}^{n} \left(\frac{3}{2}\right)^k$$

$$= \left(\frac{1}{3}\right)^n \left[\sum_{k=0}^{\infty} \left(\frac{3}{2}\right)^k - \sum_{k=n+1}^{\infty} \left(\frac{3}{2}\right)^k\right] = \left(\frac{1}{3}\right)^n \left[\frac{1}{1-3/2} - \left(\frac{3}{2}\right)^{n+1} \frac{1}{1-3/2}\right]$$

$$= \left(\frac{1}{3}\right)^n (-2)\left[1 - \left(\frac{3}{2}\right)^{n+1}\right] = 3\left(\frac{1}{2}\right)^n - 2\left(\frac{1}{3}\right)^n \quad \checkmark$$

Or, by transforms, $F(z) = \frac{1}{1 - \frac{1}{2}z}$
$H(z) = \frac{1}{1 - \frac{1}{3}z}$

$$G(z) = \frac{1}{(1-\frac{1}{2}z)(1-\frac{1}{3}z)} = \frac{3}{1-\frac{1}{2}z} + \frac{-2}{1-\frac{1}{3}z}$$

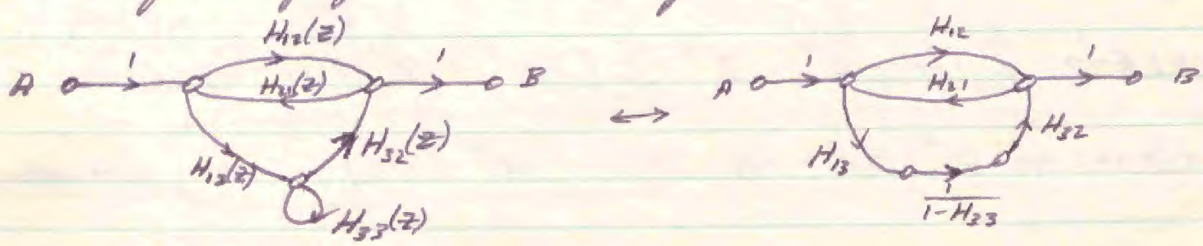
so $g(n) = 3\left(\frac{1}{2}\right)^n - 2\left(\frac{1}{3}\right)^n \quad \checkmark$

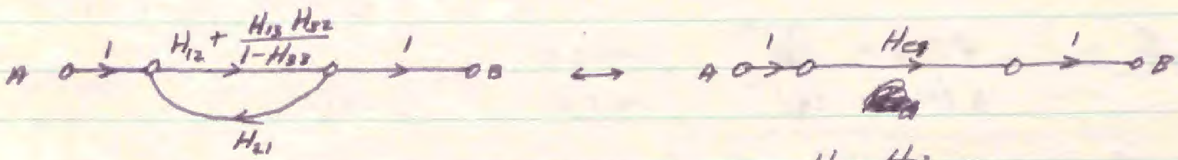
Flow graph transmission:

(1) Reduction techniques:

On page 7, we saw how we could write equivalent transfer functions for simple combinations of systems in terms of the transfer functions of the individual branches.

We can now systematically apply these reduction rules to parts of a more complex flow graph & eventually we will arrive at a single branch with the equivalent transmission of the total flow graph. For example:





$$H_{eq} = \frac{H_{12} + \frac{H_{13}H_{32}}{1-H_{33}}}{1 - H_{21}\left[H_{12} + \frac{H_{13}H_{32}}{1-H_{33}}\right]}$$

$$H_{eq} = \frac{H_{12}(1-H_{33}) + H_{13}H_{32}}{1-H_{33} - H_{12}H_{21} - H_{21}H_{13}H_{32} + H_{12}H_{21}H_{33}}$$

(2) Mason's flow graph transmission method:

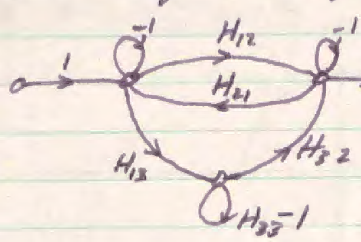
$$T_{AB} = \frac{\sum_p P_{AB}(1-L_1-L_2\cdots)^*}{(1-L_1)(1-L_2)(1-L_3)\cdots} = \frac{\sum_p P_{AB}(1-L_1-L_2\cdots)}{1-L_1-L_2-L_3+L_1L_2\cdots-L_1L_2L_3\cdots}$$

where only those products ~~in the numerator~~ are allowed for which the corresponding loops ~~and~~ &/or paths do not touch in the flow graph.

(3) <u>Coates' flow graphs and transmission</u>:

Coates' flow graph representation is slightly different than that of Mason. The signal at each node is zero (rather than some physical variable). To convert a regular flow graph to a Coates flow graph, add a (-1) to the self loop at each node. If a node has no self loop, add a loop with transmission (-1).

For example, the flow graph on the last page becomes

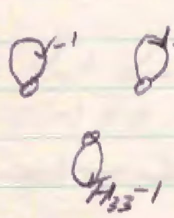


In connection with this method, a connection of the flow graph is defined:

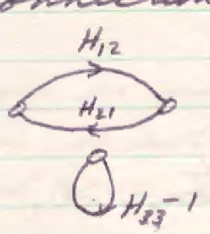
<u>connection</u> ≡ A sub-flow graph of the original with all the original nodes, but with only one input + one output branch ~~from~~ at each node: a collection of simple loops

<u>one-connection</u> ≡ same as a connection except that the input node has only one branch leaving it + the output node has only one branch entering it.

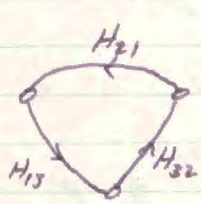
In the above example, connections, ~~and~~ with their associated gains are:



$-(H_{33}-1)$



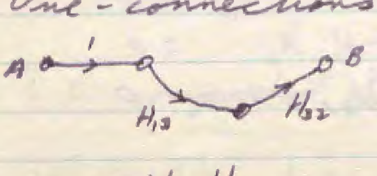
$+H_{12}H_{21}(H_{33}-1)$



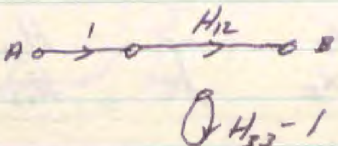
$-H_{13}H_{32}H_{21}$

Note: the sign of the connection gain is $(-1)^n$, where n is the number of loops in the connection.

One-connections are:



$+H_{13}H_{32}$



$-H_{12}(H_{33}-1)$

Now the transmission of the flow graph T_{AB} can be written directly as

$$T_{AB} = \frac{\Sigma\,(\text{one connection gains})}{\Sigma\,(\text{connection gains})}$$

Note that for transmission through different nodes, the one-connections will have to be different but the ~~denominator~~ denominator will be the same for ~~[illegible]~~ any one flow graph.

The example previously used gives

$$T_{AB} = \frac{H_{13}H_{32} + H_{12}(1 - H_{33})}{1 - H_{33} + H_{12}H_{21}H_{33} - H_{12}H_{21} - H_{13}H_{32}H_{21}}$$

The principal advantage of the Coates' method is that it eliminates all arithmetic cancelling within the denominator & between the numerator & denominator.

A flow graph is merely a graphic way to display the relation between several variables (i.e., a set of linear equations)

$$\sum_i a_{ij}\,x_i - b_j = 0 \quad , \quad i = 1, 2, \ldots, k$$

or, in matrix form:

$$A\underline{x} - \underline{b} = 0$$

In Coates flow graph representation, a_{ij} is the value of the branch from j to i : $i \overset{a_{ij}}{\longleftarrow} j$

In our example:

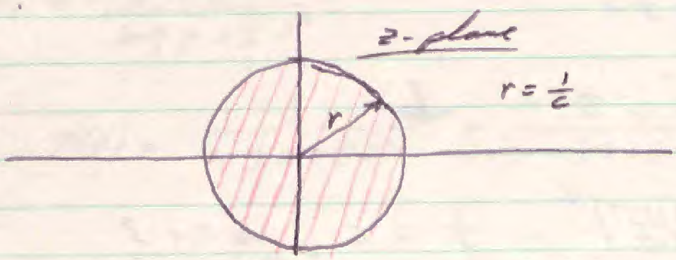
$$\left.\begin{aligned} x_1 &= H_{21}x_2 + 1 \\ x_2 &= H_{12}x_1 + H_{32}x_3 \\ x_3 &= H_{13}x_1 + H_{33}x_3 \end{aligned}\right\} \qquad \begin{aligned} -x_1 + H_{12}x_2 \qquad\quad - 1 &= 0 \\ H_{12}x_1 - x_2 + H_{32}x_3 \quad &= 0 \\ H_{13}x_1 \qquad + (H_{33} - 1)x_3 &= 0 \end{aligned}$$

Existence of transforms :

In order for the transform, $F(z) = \sum_{n=0}^{\infty} f(n) z^n$, to exist, we ~~m~~ require that $f(n)$ rise no faster than geometrically with $\underline{n}$; i.e. :

$$\lim_{n\to\infty} |f(n)|^{\frac{1}{n}} = \text{const} = c$$

If this requirement is satisfied, $F(z)$ will converge for $|z| < r = \frac{1}{c}$:



We can regain $f(n)$ by a contour integration in the z-plane:

$$f(n) = \frac{1}{2\pi i} \oint_{\substack{\text{around} \\ z=0}} \frac{F(z)\,dz}{z^{n+1}}$$

Or, if $z=0$ is included in the region of convergence, we can regard $F(z)$ as expanded in a power series, and write:

$$f(n) = \frac{1}{n!} \left. \frac{d^n F(z)}{dz^n} \right|_{z=0}$$

Note that this will always be valid if $f(n) = 0$ for $n < 0$. If $f(n) \neq 0$, $n < 0$, then we will have terms of $\frac{1}{z^k}$ in the transform and the point $z=0$ is no longer analytic.

Initial & final value theorems :

$$F(z) = \sum_{n=0}^{\infty} f(n) z^n \implies \boxed{F(0) = f(0) \text{ ; initial value theorem}}$$

The DC component of $f(n)$ can be found by

$$\boxed{\text{Res}\{F(z);\, z=1\} = \lim_{z\to 1} (1-z) F(z) = \text{avg } f(\infty) \text{ ; "final value theorem"}}$$

Finding $f(n)$ from $F(z)$:

(1) Partial fraction expansion
(2) Power series expansion (by identification or long division)
(3) differentiation

Example: $$F(z) = \frac{1}{(1-\frac{4}{3}z+\frac{1}{3}z^2)} = \frac{1}{(1-\frac{1}{3}z)(1-z)}, \quad n \geq 0$$

(1) $$F(z) = \frac{\frac{3}{2}}{1-z} + \frac{-\frac{1}{2}}{1-\frac{1}{3}z}$$

$$\frac{1}{1-\frac{1}{3}z} \leftrightarrow f(n) = \left(\tfrac{1}{3}\right)^n, \quad n \geq 0 \quad \cancel{\lim}$$

$$\lim_{n\to\infty} |f(n)|^{1/n} = \lim_{n\to\infty} \left|\left(\tfrac{1}{3}\right)^n\right|^{1/n} = \tfrac{1}{3} = c \quad \Rightarrow r = 3$$

so $\frac{1}{1-\frac{1}{3}z}$ converges for $|z| < 3$.

Similarly, $\frac{1}{1-z}$ converges for $|z| < 1$

so $F(z)$ converges for $|z|<1$.

$$f(n) = \frac{3}{2} - \frac{1}{2}\left(\frac{1}{3}\right)^n \qquad \begin{cases} f(0) = 1 \\ f(1) = \frac{4}{3} \\ f(2) = \frac{13}{9} \end{cases}$$

(2)
$$\begin{array}{rl} & 1 + \frac{4}{3}z + \frac{13}{9}z^2 + \cdots \\ 1 - \frac{4}{3}z + \frac{1}{3}z^2 \,) & 1 \\ & \underline{1 - \frac{4}{3}z + \frac{1}{3}z^2} \\ & \frac{4}{3}z - \frac{1}{3}z^2 \\ & \underline{\frac{4}{3}z - \frac{16}{9}z^2 + \frac{4}{9}z^3} \\ & \frac{13}{9}z^2 + \frac{4}{9}z^3 \end{array}$$

$$F(z) = 1 + \frac{4}{3}z + \frac{13}{9}z^2 + \cdots$$

$f(0) = 1$
$f(1) = \frac{4}{3}$
$f(2) = \frac{13}{9}$

(3) $f(0) = F(0) = 1$

$$f(1) = \left.\frac{dF(z)}{dz}\right|_{z=0} = \left.\frac{-1\left[-\frac{4}{3}+\frac{2}{3}z\right]}{\left(1-\frac{4}{3}z+\frac{1}{3}z^2\right)^2}\right|_{z=0} = +\frac{4}{3}$$

. – – – —

Finding $f(n)$ when ~~the~~ $F(z)$ has a higher power of z in the numerator:

e.g. $$F(z) = \frac{z^2+2}{1-\frac{4}{3}z+\frac{1}{3}z^2}$$

(1) Write $$F(z) = \frac{2}{1-\frac{4}{3}z+\frac{1}{3}z^2} + \frac{z^2}{1-\frac{4}{3}z+\frac{1}{3}z^2}$$

then
$$f(n) = \underbrace{\left[3-\left(\tfrac{1}{3}\right)^n\right]}_{n\geq 0} + \underbrace{\left[\tfrac{3}{2}-\tfrac{1}{2}\left(\tfrac{1}{3}\right)^{n-2}\right]}_{n\geq 2}$$

An alternative approach is to use long division with remainder:

$$\begin{array}{r|l} & \;3 \\ \hline \frac{1}{3}z^2-\frac{4}{3}z+1 & z^2+0+2 \\ & z^2-4z+3 \\ \hline & 4z-1 \end{array} \Bigg\} \Rightarrow F(z) = 3 + \frac{4z-1}{1-\frac{4}{3}z+\frac{1}{3}z^2}$$

$$F(z) = 3 + \frac{9/2}{1-z} + \frac{-11/2}{1-\frac{1}{3}z}$$

$$\Rightarrow f(n) = 3\delta(n) + \frac{9}{2} - \frac{11}{2}\left(\frac{1}{3}\right)^n, \quad n\geq 0$$

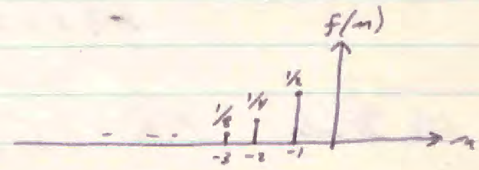
Transforms & signals for negative time, $n<0$:

Suppose we have defined a signal $f(n)$ for $n<0$. Then a transform analogous to the situation for $n \geq 0$:

$$F(z) = \sum_{n=-1}^{-\infty} f(n)\, z^{n}$$

This transform is a series of powers of $\left(\frac{1}{z}\right)$ and can be expanded about the point $z=-\infty$. It converges outside a circle $|z| = r$ in the complex z-plane.

For example, suppose $f(n) = 2^{n}$, $n<0$



$$F(z) = \sum_{n=-1}^{-\infty} f(n)\, z^{n} = \sum_{n=-1}^{-\infty} (2z)^{n} = \sum_{k=1}^{\infty} \left(\frac{1}{2z}\right)^{k}$$

$$F(z) = \frac{\frac{1}{2z}}{1-\frac{1}{2z}} = \frac{1}{2z} + \frac{1}{4z} + \frac{1}{8z^{3}} + \cdots \; ; \; \underline{\text{converges for } |z| > \tfrac{1}{2}}$$

Now, we know that

$$\underbrace{F(z) = \frac{1}{1-az} \leftrightarrow f(n) = a^{n}, \quad n \geq 0.} \quad \left[\underbrace{F(z) \text{ converges } |z| < \tfrac{1}{a}}\right]$$

But we can re-write $F(z)$ as

$$\underbrace{F(z) = \frac{1}{1-az} = \frac{-\frac{1}{az}}{1-\frac{1}{az}} \leftrightarrow f(n) = -a^{n}, \quad n<0.} \quad \left[F(z) \text{ converges } |z| > \tfrac{1}{a}\right]$$

Thus we see that the transform of a time function really has two parts: the functional form plus the region of convergence. The latter is determined by the range over which $f(n)$ is valid.

There will be no ambiguity if both $F(z)$ and the region of convergence are specified.

Two-sided transforms:

We extend our earlier ideas about transforms to the case where $f(n)$ is defined over the doubly infinite interval $(-\infty, \infty)$:

$$\underline{F(z) = \sum_{n=-\infty}^{\infty} f(n) z^{-n}} = \underbrace{\sum_{n=-\infty}^{-1} f(n) z^{-n}}_{\text{descending part}} + \underbrace{\sum_{n=0}^{\infty} f(n) z^{-n}}_{\text{ascending part.}}$$

From the preceding discussions, we see that the descending part of the double-ended transform will converge outside the circle $|z| > a$. Similarly, the ascending part will converge for $|z| < b$. Thus the two-sided transform will in general converge inside a ring in the complex z-plane:

$$\underline{a < |z| < b}$$

$$f(n) = \frac{1}{2\pi i} \oint_{\substack{\text{in ring} \\ \text{about origin}}} \frac{F(z)\,dz}{z^{n+1}}$$

We cannot now differentiate to find $f(n)$ from $F(z)$ since $F(z)$ is no longer analytic about $z = 0$.

<u>Example</u>: $F(z) = \dfrac{-z}{2z^2 - 3z + 1}$, $\left\{ \begin{matrix} f(n) \text{ is bounded} \\ \lim_{n\to-\infty} f(n) = 0 \end{matrix} \right\}$; find $f(n)$:

$$F(z) = \frac{-z}{(1-z)(1-2z)} = \frac{1}{1-z} + \frac{-1}{1-2z}$$

Now, $f(n)$ is bounded, so $\frac{1}{1-2z}$ must be valid only for $n < 0$ since 2^n blows up for $n \to \infty$. Similarly, $\frac{1}{1-z}$ can be valid only for positive n since the step would not die out for large negative n: Thus:

$$\left. F(z) = \underset{n \geq 0}{\frac{1}{1-z}} + \underset{n<0}{\frac{\frac{1}{2z}}{1 - \frac{1}{2z}}} \right\} \text{ converges } \tfrac{1}{2} < |z| < 1$$

$$\underline{f(n) = \left\{ \begin{matrix} 1, & n \geq 0 \\ 2^n, & n < 0 \end{matrix} \right\}}$$

Laplace and z-transforms:

The z-transform can be shown to be a special case of the Laplace transform when the time signal is a "continuous time function made up entirely of impulses."

Let $f(t) = \sum_n f(n)\,\delta(t-n)$

$$F(s) = \int f(t) e^{-st}\,dt = \int \sum_n f(n)\,\delta(t-n) e^{-st}\,dt$$

$$= \sum_n f(n) \int \delta(t-n) e^{-st}\,dt = \sum_n f(n) e^{-sn}$$

If we let $Z = e^{-s}$, or $s = -\ln z$

$F(z) = \sum_n f(n) z^n$: the z-transform.

The inverse transform can be found similarly:

$$f(t) = \frac{1}{2\pi j} \oint F(s) e^{st}\,ds = \frac{1}{2\pi j} \oint \sum_n f(n) e^{-sn} e^{st}\,ds$$

$$= \frac{1}{2\pi j} \oint F(z) \left(\frac{1}{z}\right)^t d(-\ln z)$$

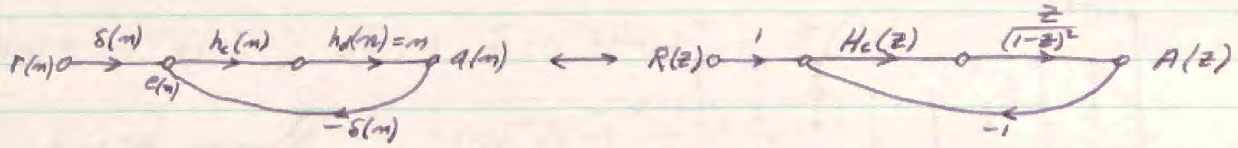
But t can only take on values $0, 1, 2, \ldots$

$$f(n) = \frac{1}{2\pi j} \oint F(z) \frac{1}{z^n} \frac{dz}{z} = \frac{1}{2\pi j} \oint \frac{F(z)}{z^{n+1}}\,dz$$

as stated on page 13.

Example: The one-arm driver:

The roadway position is a given input $r(n)$. The auto position is $a(n)$, determined by how we steer in response to $e(n)$. The dynamic response of the car to an impulse in steering is a ramp $h_d(n) = n$, $n \geq 0$. The error, $r(n) - a(n)$, is $e(n)$:



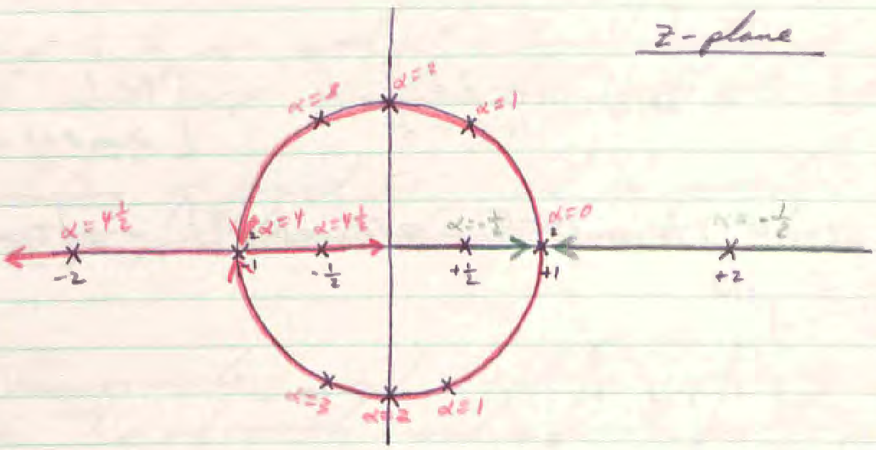
Suppose we let $h_c(n) = \alpha\,\delta(n)$ be the response to the error $e(n)$:
$H_c(z) = \alpha$

$$T_{RA}(z) = \frac{\dfrac{\alpha z}{(1-z)^2}}{1 + \dfrac{\alpha z}{(1-z)^2}} = \frac{\alpha z}{1-(2-\alpha)z+z^2}$$

The roots of the denominator are $z_{1,2} = \dfrac{\alpha - 2}{2} \pm \frac{1}{2}\sqrt{\alpha(\alpha-4)}$

We can now make a table of these roots for various values of α.

α	z_1, z_2
$-\frac{1}{2}$	$\frac{1}{2}, 2$
0	$1, 1$
1	$\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$
2	$\pm j$
3	$-\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$
4	$-1, -1$
$4\frac{1}{2}$	$-\frac{1}{2}, -2$



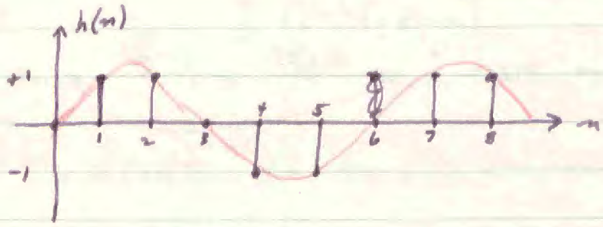
Poles inside the unit circle will blow up; poles outside will die down. What will the complex poles do?

$\alpha = 0$ is a useless system since it gives $T_{RA}(z) = 0$. So we look at $\alpha = 1$:

$\underline{\alpha = 1}$: $T_{RA}(z) = \dfrac{z}{1 - z + z^2} \Longleftrightarrow \dfrac{z r \sin\theta}{1 - 2rz\cos\theta + r^2 z^2} \leftrightarrow \underline{f(n) = r^n \sin n\theta}$

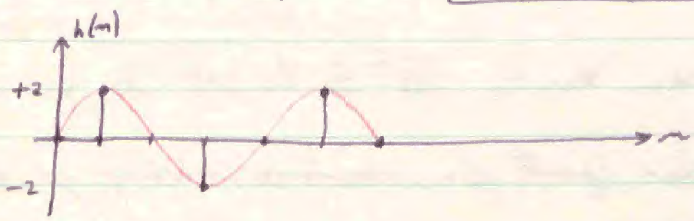
We can identify terms in these two transforms by letting $r = 1$, $\cos\theta = +\frac{1}{2} \Rightarrow \theta = \pi/3 \Rightarrow \sin\theta = \frac{\sqrt{3}}{2}$. Letting $R(z) = 1$,

$$h(n) = \frac{2}{\sqrt{3}} \sin\left(\frac{n\pi}{3}\right) \leftrightarrow \boxed{\text{period } T = \frac{2\pi}{\pi/3} = 6}$$



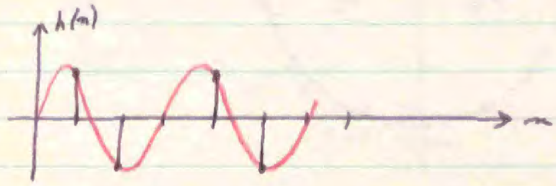
$\underline{\alpha = 2}$: $T_{RA}(z) = \dfrac{2z}{1 + z^2}$ $\quad \begin{cases} r = 1 \\ \cos\theta = 0 \Rightarrow \theta = \pi/2 \Rightarrow \sin\theta = +1 \end{cases}$

$$h(n) = 2 \sin\left(\frac{n\pi}{2}\right) \leftrightarrow \boxed{T = \frac{2\pi}{\pi/2} = 4}$$



$\underline{\alpha = 3}$: $T_{RA}(z) = \dfrac{3z}{1 + z + z^2}$ $\quad \begin{cases} r = 1 \\ -2\cos\theta = 1 \Rightarrow \theta = \frac{2\pi}{3} \Rightarrow \sin\theta = +\frac{\sqrt{3}}{2} \end{cases}$

$$h(n) = 3\,\frac{2}{\sqrt{3}} \sin\left(\frac{2n\pi}{3}\right) = 2\sqrt{3} \sin\left(\frac{2n\pi}{3}\right) \leftrightarrow \boxed{T = \frac{2\pi}{2/3\,\pi} = 3}$$



$\underline{\alpha = 4}$: $T_{RA}(z) = \dfrac{4z}{1+2z+z^2} = \dfrac{4z}{(1+z)^2}$

Now, we have to find a new way to find the inverse transform here:

$F(z) = \sum f(n) z^n = \sum (-1)^n f(n) (-z)^n \leftrightarrow f(n)$

If $g(n) = (-1)^n f(n)$, $G(z) = \sum (-1)^n f(n) z^n = F(-z)$

So, we have the result $\boxed{(-1)^n f(n) \leftrightarrow F(-z)}$

Thus $\dfrac{4z}{(1+z)^2} = F(-z) \Rightarrow F(z) = \dfrac{-4z}{(1-z)^2} \leftrightarrow -4n$, $n \geq 0$

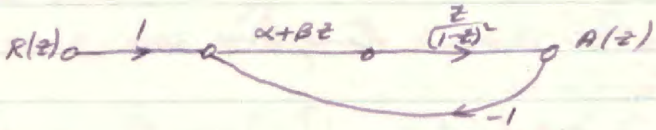
so here, $\underline{h(n) = -4(-1)^n n = 4(-1)^{n+1} n}$ $\leftrightarrow T = 2$?
but blows up.

$\underline{\alpha = -\frac{1}{2}}$: With the real poles at $\frac{1}{2}$, 2 we will get a time function of the form $A 2^n + B\left(\frac{1}{2}\right)^n$ with no periodicity which will blow up.

$\underline{\alpha = 4\frac{1}{2}}$: We will get "oscillations" of period $T=2$ but the envelope
$\alpha = 5\frac{1}{2}$: will blow up due to the pole inside the unit circle.
⋮

Note that none of these responses to a unit impulse in roadway position die away in time. We obviously need a better system, i.e., a better compensation response $h_c(n)$.

Now try $H_c(z) = \alpha + \beta z$. That is our correction is for both present error and error ~~due~~ at the last time step. Our system is now:



$$T_{RA}(z) = \frac{(\alpha+\beta z)z}{1-(2-\alpha)z+(1+\beta)z^2}$$

$$z_1, z_2 = \frac{1}{2(1+\beta)}\left[2-\alpha \pm \sqrt{\alpha^2-4(\alpha+\beta)}\right]$$

We now want to look at the $\alpha\beta$ plane to find values of α, β for stability.

For complex roots, both roots must lie ~~on~~ outside the unit circle in the complex z-plane for stability: If the denominator is $ax^2+bx+c=0$ $= (x-x_1)(x-x_2)$, we see $\left|\frac{c}{a}\right| = x_1 x_2 = x_1 x_1^* = |x_1|^2 \geq 1$ ~~[illegible]~~. Here,

$$\left|\left(\frac{c}{a}\right)\right| = \left|\frac{1}{1+\beta}\right| \geq 1 \quad \text{or} \quad |1+\beta| < 1 \quad \text{or} \quad \underline{-2 < \beta < 0 \text{ for complex roots}}$$

For real roots, the boundary between stability and instability is given by $|z_{pole}| = 1$. We now plug in $z=+1$ and $z=-1$ and find what conditions on α and β ~~give~~ make the denominator of $T_{RA}(z) = 0$; i.e., make ~~a the poles~~ $z = \pm 1$ a pole of $T_{RA}(z)$.

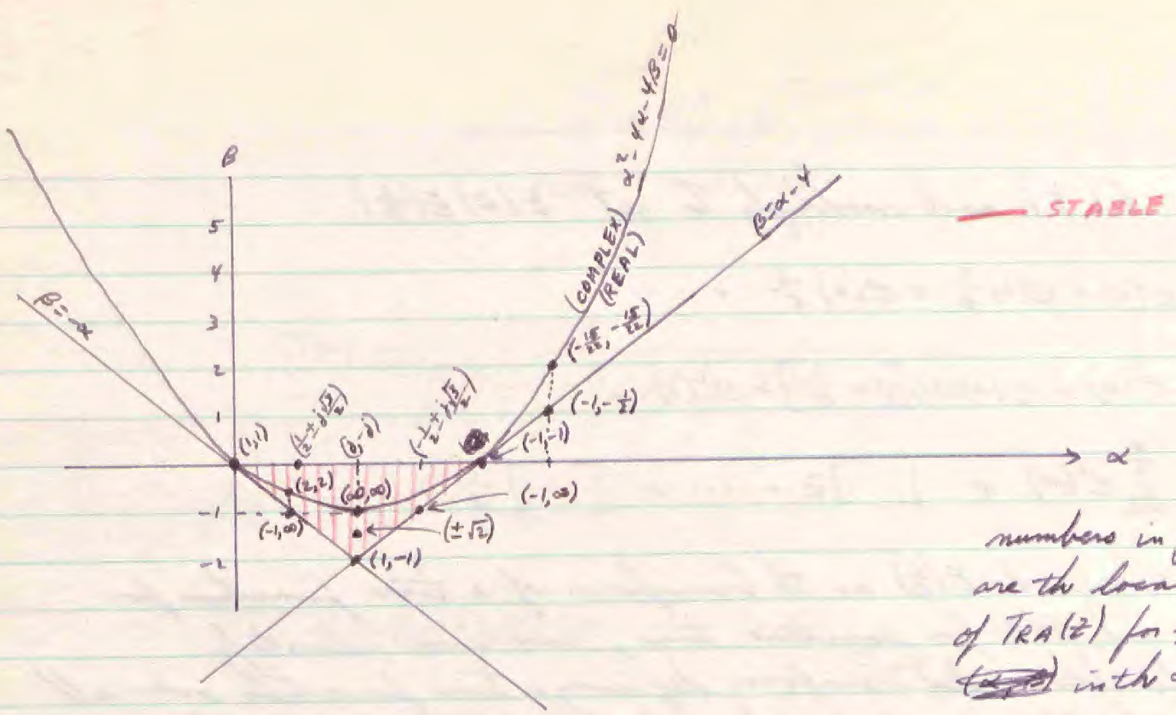
$z=+1$, $\quad 1-(2-\alpha)z+(1+\beta)z^2 = 0 = 1-2+\alpha+1+\beta = \alpha+\beta$
$\Rightarrow \underline{\alpha = -\beta}$

$z=-1$, $\quad 0 = 1+(2-\alpha)+(1+\beta) = 4-\alpha+\beta \Rightarrow \underline{\beta = \alpha - 4}$

These two lines divide the $\alpha\beta$ plane into four regions, in <u>one</u> of these regions, <u>both</u> poles of $T_{RA}(z)$ will lie outside the unit circle. (In the others, one or both will lie inside.)

We will get complex roots of $T_{RA}(z)$ when

$\alpha^2 - 4(\alpha+\beta) < 0$ or $\alpha^2 - 4\alpha - 4\beta = 0$ is the boundary between real + complex roots.



numbers in parentheses are the locations of the poles of $T_{RA}(z)$ for the point indicated in the $\alpha\beta$ plane.

Now, look at the error response of the system:

$$T_{RE}(z) = \frac{(1-z)^2}{1-(2-\alpha)z+(1+\beta)z^2}$$

If we pick α and β for a stable system, the error for an impulse, a step, or a ramp will eventually die away since the $(1-z)^2$ term in the numerator of T_{RE} will cancel the $(1-z)$ terms in the denominators of $R(z)$, so no step component remains in the error response. For a parabolic input though, where $R(z) = \frac{z(1+z)}{(1-z)^3}$, there will be a $\frac{1}{1-z}$ term in $E(z)$ and the error would never die out. Thus this system would be lousy in a situation where parabolic inputs are likely to occur.

"Summed-square-error" as performance index:

By analogy to integral square error in continuous systems, we might be prompted to use $\sum_{n=0}^{\infty} e^2(n) = E$ as a measure of how well our system performs. We can evaluate this as follows. Transform the error signal $e(n)$ to get

$$E(z) = e(0) + e(1)z + e(2)z^2 + \cdots$$

Now find $E(\frac{1}{z})$ and multiply to get $E(z)E(\frac{1}{z})$

$$E(\tfrac{1}{z}) = e(0) + e(1)\tfrac{1}{z} + e(2)\tfrac{1}{z^2} + \cdots$$

$$E(z)E(\tfrac{1}{z}) = e^2(0) + e(0)e(1)[z + \tfrac{1}{z}] + e^2(1) + \cdots$$

$$= \sum_{n=0}^{\infty} e^2(n) + [\quad]z + \cdots + [\quad]\tfrac{1}{z} + \cdots$$

If we now regard $E(\frac{1}{z})$ as the transform of a time function for negative time and $E(z)$ for positive time, we can identify $E(z)E(\frac{1}{z})$ as a double-sided transform defining a time function over all time. By finding this time function and evaluating the term for $z=0$, we can find the total squared error.

<u>Example</u>: In our preceding system, $T_{RE}(z) = \dfrac{(1-z)^2}{1-(2-\alpha)z+(1+\beta)z^2}$

For a ramp input, $R(z) = \dfrac{z}{(1-z)^2} \Rightarrow E(z) = \dfrac{z}{1-(2-\alpha)z+(1+\beta)z^2}$

$$E(\tfrac{1}{z}) = \frac{z^{-1}}{1-(2-\alpha)z^{-1}+(1+\beta)z^{-2}} = \frac{z}{z^2-(2-\alpha)z+(1+\beta)}$$

$$E(z)E(\tfrac{1}{z}) = \frac{z^2}{[1-(2-\alpha)z+(1+\beta)z^2][z^2-(2-\alpha)z+(1+\beta)]} = \frac{Az+B}{[1-(2-\alpha)z+(1+\beta)z^2]} + \frac{Cz+D}{[z^2-(2-\alpha)z+(1+\beta)]}$$

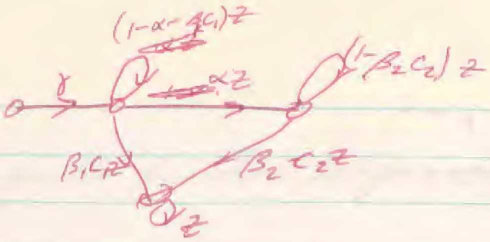
$$= \frac{Az+B}{[1-(2-\alpha)z+(1+\beta)z^2]} + \frac{C\frac{1}{z}+D\frac{1}{z^2}}{[1-(2-\alpha)\frac{1}{z}+(1+\beta)\frac{1}{z^2}]}$$

$$= \frac{Az+B}{[1-(2-\alpha)z+(1+\beta)z^2]} + \frac{\frac{1}{z}[C+D\frac{1}{z}]}{[1-(2-\alpha)\frac{1}{z}+(1+\beta)\frac{1}{z^2}]}$$

We can now identify the first term as corresponding to $n \geq 0$ and the second term for $n < 0$, since $(\frac{1}{z})$ factors out of the second term, insuring that it contributes nothing for $n = 0$.

Thus the central term is

$$\sum_{n=0}^{\infty} e^2(n) = \frac{B(\alpha,\beta)}{[1-(2-\alpha)z+(1+\beta)z^2]_{z=0}} = B(\alpha,\beta)$$



$$T(z) = \frac{\gamma \alpha z}{[1-(1-\alpha-\beta_1 c_1)z][1-(1-\beta_2 c_2)z]}$$

$$T(1) = \frac{\alpha \gamma}{(\alpha+\beta_1 c_1)(\beta_2 c_2)}$$

~~$m_s(t) = m(z)\, m(\frac{1}{z})$~~

$$\cancel{\frac{1}{(1-z)(1-\frac{1}{z})} = \frac{z}{(1-z)^2}}$$

$$E(z) = m(z)\, m(\tfrac{1}{z})$$

$$\frac{1}{1-z} + \frac{1}{1-\frac{1}{z}}$$

~~$\{m(1)\}^2$~~

$$-\frac{1}{z} + z$$

$$E(z) = G(z) G(\tfrac{1}{z}) T(z) T(\tfrac{1}{z}) =$$

Let $G(z) = \frac{A}{1-z}$

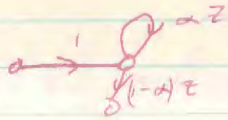
$$G(z) G(\tfrac{1}{z}) = \frac{A^2}{(1-z)(1-\frac{1}{z})} = \frac{-A^2 z}{(1-z)^2}$$

$$T(z) = \frac{\gamma \alpha z}{(1-\beta_1 z)(1-\beta_2 z)}$$

$$E(z) = \frac{-A^2 z}{(1-z)^2} \frac{\gamma^2 \alpha^2}{(1-\beta_1 z)(1-\beta_1 \frac{1}{z})(1-\beta_2 z)(1-\beta_2 \frac{1}{z})}$$

$$E(1) = \frac{-A^2 \gamma^2 \alpha^2}{}$$

Is min $E(1) \Longleftrightarrow$ min SS level?

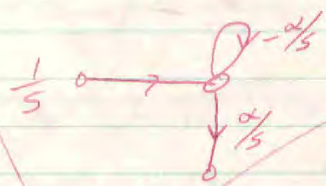


$$T(z) = \frac{1}{1-\alpha z}$$

$$M(z) = \frac{T(z)}{1-\alpha z}$$

$$(1-z)\,M(1) = \frac{1}{1-\alpha}$$

$$E(z) = M(z)M(\tfrac{1}{z}) = \frac{1}{(1-z)(1-\frac{1}{z})(1-\alpha z)(1-\alpha\frac{1}{z})} = \frac{z^2}{(1-z)^2(1-\alpha z)(\alpha - z)}$$



$$T(s) = \frac{\frac{1}{s}}{1+\frac{\alpha}{s}} = \frac{1}{s+\alpha}.$$

$$M(s) = \frac{1}{s}T(s) = \frac{1}{s(s+\alpha)}$$

$$M(s)M(-s) = \frac{1}{-s^2(s+\alpha)(\alpha-s)}$$

$$E(z) = \frac{Az+B}{(1-z)(1-\alpha z)} + \frac{Cz+D}{(1-z)(\alpha-z)} = \frac{z^2}{(1-z)^2(1-\alpha z)(\alpha-z)}$$

$$= \underbrace{\frac{Az+B}{(1-z)(1-\alpha z)}}_{t\geq 0} + \underbrace{\frac{\frac{1}{z}\left(C+\frac{D}{z}\right)}{(1-\frac{1}{z})(1-\frac{\alpha}{z})}}_{t<0}$$

$$\boxed{A\left(\alpha-\tfrac{1}{\alpha}\right) = B\left(\alpha-\tfrac{1}{\alpha}\right) \quad A = B}$$

$$E(0) = B$$

$$(Az+B)(1-z)(\alpha-z) + (Cz+D)(1-z)(1-\alpha z) = z^2$$

$$= (Az+B)(\alpha-\alpha z - z + z^2) + (Cz+D)(1-z-\alpha z+\alpha z^2) = z^2$$

$$0 = \alpha B + D \Rightarrow D = -B/\alpha$$

$$0 = A\alpha - B\alpha - B + C - D - \alpha D = A\alpha - B\alpha \cancel{-B} - \frac{A}{\alpha} + B/\alpha \cancel{+B}$$

$$1 = B - A\alpha - A + D\alpha - C - \alpha C \rightarrow$$

$$0 = A + C\alpha \Rightarrow C = -A/\alpha$$

$$1 = \cancel{B} - B\alpha \cancel{-B} \cancel{-B} + B/\alpha \cancel{+B}$$

$$1 = B\left(\tfrac{1}{\alpha}-\alpha\right) \Rightarrow B = \frac{1}{\frac{1}{\alpha}-\alpha} = \frac{\alpha}{1-\alpha^2}$$

If we carry out the partial fraction expansion far enough to evaluate B, we find

$$B = \frac{1-(1+\beta)^2}{[1-(1+\beta)^2]^2-(2-\alpha)^2\beta^2} \quad , \ \underline{\alpha, \beta \text{ in region of stability}}$$

We now want $\min\limits_{\alpha,\beta} B$.

We can minimize ~~B~~ over α by simply maximizing the denominator over α, or, minimizing $(2-\alpha)^2$ over $\alpha \Rightarrow \underline{\alpha = +2}$.

Now, $\min\limits_{\beta} \dfrac{1-(1+\beta)^2}{[1-(1+\beta)^2]^2} = \min\limits_{\beta} \dfrac{1}{1-(1+\beta)^2}$

This is clearly a minimum if $(1+\beta)^2$ is a minimum or if $\beta = -1$.

Then $(\alpha = 2, \beta = -1)$ gives minimum total square error

$$\left[\sum_{n=0}^{\infty} e^2(n)\right]_{min} = 1.$$

This system is then "optimal" for a ramp input. It is <u>not</u> optimal for a step or impulse input.

<u>Brief review of matrices</u>:

$(x_1, \ldots, x_m) \longrightarrow (y_1, \ldots, y_m)$ is equivalent to $y_j = \sum_i a_{ji} x_i$

In matrix notation,

$$x] = \begin{matrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{matrix}\Bigg]$$

$$(a_{ij})\, x] = y] \equiv Ax] = y]$$

A is a linear operator by virtue of our definition $y_j = \sum a_{ji} x_i$

There exists a solution $x]$ to $A x] = y]$ $\Leftrightarrow$ $\det A \neq 0$
Then $\exists$ $A^{-1} \ni$ $A^{-1} y] = x]$, $AA^{-1} = I = A^{-1}A$

If $z] = BAx] = By]$, $z_k = \sum_i b_{ki} y_i = \sum_i b_{ki} \sum_j a_{ij} x_j$

$$z_k = \sum_j \left[\sum_i b_{ki} a_{ij} \right] x_j = \sum_j c_{kj} x_j$$

$$\underline{z] = C x] \quad , \quad C = BA}$$

Eigenvalues:
Suppose $Ax = \lambda x = \lambda I x$
This equation is equivalent to

$$[A - \lambda I] x = 0$$

Thus there will be a solution for any given λ only if $\det[A - \lambda I] = 0$.

$|A - \lambda I| = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_N)$

We will consider only the case when $N = n$; there are n distinct non-equal values of λ. Now if we pick λ to be one of these roots, say $\lambda = \lambda_k$. Then $\det(A - \lambda I) = 0$ and the rank of $(A - \lambda I)$ is $n-1$. Thus the values we find for $x]$ with this value of λ is determined only within a constant multiplier.

Suppose we now find solutions $x]_1, \ldots, x]_n$ for each value of λ. The resulting vectors are called eigenvectors and the λ_i are called eigenvalues.

Now, no non-trivial linear combination of these eigenvectors is identically zero. That is, the eigenvectors are independent (linearly).

Now, from these column vectors, form the square matrix

$$E \equiv \left(x]_1 \; x]_2 \cdots x]_n \right)$$

Consider $E^{-1}AE$

$$AE = A(x]_1, \cdots x]_m) = (\lambda_1 x]_1, \cdots \lambda_m x]_m)$$

$$E^{-1}AE = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \lambda_m \end{pmatrix}$$

Now $E^{-1}AE$ is a diagonal matrix, $E^{-1}AE = D$, so $D^k = \begin{pmatrix} \lambda_1^k & & 0 \\ & \lambda_2^k & \\ 0 & & \ddots \lambda_m^k \end{pmatrix}$

$$A^k = E[E^{-1}AEE^{-1}AE\cdots]E^{-1} = EE^{-1}A^kEE^{-1} = ED^kE^{-1}$$

$$\boxed{\begin{aligned} A = EDE^{-1} &= \lambda_1 E\begin{pmatrix} 1&0&0\\0&0&0\\0&0&0 \end{pmatrix}E^{-1} + \lambda_2 E\begin{pmatrix} 0&0&0\\0&1&0\\0&0&0 \end{pmatrix}E^{-1} + \cdots \\ &= \lambda_1 P_1 + \lambda_2 P_2 + \cdots \\ \text{Note} \quad P_i P_j &= \delta_{ij} P_i \end{aligned}}$$

$$P_i\, z] = x_1] \times \underbrace{y_1 \cdot z}$$

so $A z] = \lambda_1 x_1] \times \underbrace{y_1 \cdot z} + \lambda_2 x_2] \times \underbrace{y_2 \cdot z} + \cdots$

Analog: Time functions as vectors:

In the past pages we have studied linear operators of the form $y_i = \sum_{i=1}^{N} a_{ij} x_j$. We would now like to extend this to an infinite-dimensional vector space so we can similarly treat ~~the~~ time functions $\underline{f}$ as vectors with components $f(n)$. Here linear operators of interest operate by convolution: $g(n) = \sum h(n-k) f(k)$.
Or, in continuous time, $g(t) = \int h(t-\tau) f(\tau)$.

Eigenvalues & eigenfunctions:

Substituting the word "function" for "vector", we ask if there is some function $\underline{f}$ which yields $\lambda \underline{f}$ when operated on by a linear operator $h(\cdot)$: i.e.,

$$\mathcal{H}[f(n)] = \sum_k f(k) h(n-k) = \lambda f(n)$$

Suppose $f(n) = X^{-n}$; then

$$\sum_k X^{-k} h(n-k) = X^{-n} \sum_{k=-\infty}^{n} X^{n-k} h(n-k) = X^{-n} \sum_{k=0}^{\infty} X^{k} h(k)$$

Now $\lambda [X^{-n}] = X^{-n} \sum_{k=0}^{\infty} X^k h(k) \Rightarrow \lambda = \sum_{k=0}^{\infty} X^k h(k) = F(z)\Big|_{z=x} = F(x)$

Thus, $f(n) = a^{-n}$ is an eigenfunction of the linear operator and its eigenvalue is $\lambda = F(a)$. This holds for any a in the region of convergence of $F(z)$.

For a continuous time function, ~~the~~ the equation that an eigenfunction must satisfy is

$$\int_{-\infty}^{t} f(\tau) h(t-\tau) d\tau = \lambda f(t)$$

If $f(t) = e^{st}$, $\int_{-\infty}^{t} e^{s\tau} h(t-\tau) d\tau = e^{st} \int_{-\infty}^{t} h(t-\tau) e^{-s(t-\tau)} d\tau$

or $\lambda e^{st} = e^{st} \int_0^{\infty} h(t) e^{-st} dt = H(s) e^{st}$

Thus the eigenfunction e^{st} has an eigenvalue $\lambda_s = H(s)$

Note that for finite dimensional vectors we got a finite number of eigenvectors. For discrete functions, we got an infinite number of eigenfunctions, each eigenfunction corresponding to a point in a finite region of the complex ~~a~~ z-plane. For continuous time functions, we got an infinite number of eigenfunctions, each corresponding to a point in an infinite strip of the s-plane.

Transforms from matrix analogs:

Recall for the finite vector case $A\,x] = y]$, the operation could be written as

$$(E^{-1}AE)(E^{-1}x]) = E^{-1}y]$$

or $\quad D\,\mathrm{X}] = \mathrm{Y} \quad$ where $\quad \mathrm{X}] = E^{-1}x] \;\;\&\;\; \mathrm{Y}] = E^{-1}y]$

Now, using this latter form, we can write

$$D = \begin{bmatrix} H(s_1) & 0 & 0 & 0 \\ & H(s_2) & & \\ 0 & & \ddots & \end{bmatrix}, \quad \mathrm{X}] = \begin{bmatrix} \mathrm{X}(s_1) \\ \mathrm{X}(s_2) \\ \vdots \end{bmatrix}, \quad \mathrm{Y}] = \begin{bmatrix} \mathrm{Y}(s_1) \\ \mathrm{Y}(s_2) \\ \vdots \end{bmatrix}$$

from our experience with finite vectors. Thus:

$$\boxed{D\,\mathrm{X}] \rightarrow H(s)\mathrm{X}(s) = \mathrm{Y}(s)}$$

Now $\quad (EDE^{-1})(E\mathrm{X}]) = E\mathrm{Y}]$

But
$$E = \left(\begin{bmatrix} e^{s_1 t_1} \\ e^{s_1 t_2} \\ \vdots \end{bmatrix} \begin{bmatrix} e^{s_2 t_1} \\ e^{s_2 t_2} \\ \vdots \end{bmatrix} \cdots \right)$$

so $\quad E\mathrm{X}] \rightarrow \int \mathrm{X}(s)e^{st}\,ds \equiv x(t) \leftarrow \begin{bmatrix} x(t_1) \\ x(t_2) \\ \vdots \end{bmatrix}$

$$E\mathrm{Y}] \rightarrow \int \mathrm{Y}(s)e^{st}\,ds \equiv y(t)$$

$$EDE^{-1} \rightarrow \int e^{st}H(s)e^{-s\tau}\,ds \equiv h(t-\tau)$$

Combining these, we get

$$D\,\mathbb{X} = \mathbb{Y} \;\rightarrow\; (EDE^{-1})E\mathbb{X}] = E\mathbb{Y} = A\,x] = y]$$

$$\rightarrow \begin{pmatrix} h(t_1-\tau_1) & h(t_1-\tau_2) & \cdots \\ h(t_2-\tau_1) & h(t_2-\tau_2) & \cdots \\ - & - & \cdots \end{pmatrix} \begin{bmatrix} x(\tau_1) \\ x(\tau_2) \\ \vdots \end{bmatrix} = \begin{bmatrix} y(t_1) \\ y(t_2) \\ \vdots \end{bmatrix}$$

or $\int h(t-\tau)\,x(\tau)\,d\tau = y(t)$

Summary:

If we have a problem such as $y(t) = \int h(t-\tau)\,x(\tau)\,d\tau$ to solve, we can find the eigenvectors e^{st} to transform this operation on $x(t)$ to a diagonal one: $E^{-1}x(t) = \mathbb{X}(s)$, $E^{-1}h(t)E = H(s)$
Then the transformed $y(t)$ is found by simple multiplication

$$\mathbb{Y}(s) = H(s)\,\mathbb{X}(s) = E^{-1}y(t)$$

We can now find $y(t)$ by inverse transforming:

$$y(t) = EE^{-1}y(t) = E\,\mathbb{Y}(s) = E[H(s)\,\mathbb{X}(s)]$$

$$E^{-1}x] \rightarrow \int x(t)e^{-st}\,dt = \mathbb{X}(s)$$

$$E\,\mathbb{X}] \rightarrow \int \mathbb{X}(s)e^{st}\,ds = x(t)$$

$$E^{-1}[h]E \rightarrow \int e^{-st}\,h(t-\tau)\,e^{st}\,d(t-\tau) = H(s).$$

$$E[H]E^{-1} \rightarrow \int e^{st}\,H(s)\,e^{-st}\,ds = h(t-\tau)$$

Solution of difference equations by matrix techniques:

$$g(n) = a_1\, g(n-1) + a_2\, r(n-1) + f(n)$$
$$r(n) = b_1\, g(n-1) + b_3\, r(n-1)$$

$$G(n) = \left.\begin{matrix} g(n) \\ r(n) \\ r(n-1) \end{matrix}\right] = \begin{pmatrix} a_1 & a_2 & 0 \\ b_1 & 0 & b_3 \\ 0 & 1 & 0 \end{pmatrix} \left.\begin{matrix} g(n-1) \\ r(n-1) \\ r(n-2) \end{matrix}\right] + \left.\begin{matrix} f(n) \\ 0 \\ 0 \end{matrix}\right]$$

$$G(n)] = H\, G(n-1)] + F(n)]$$

$$E^{-1} G(n)] = (E^{-1} H E)\, E^{-1} G(n-1)] + E^{-1} F(n)$$

Here it is theoretically possible but practicably ~~a~~ impossible to find the eigenvectors. Hence, we try the transform:

$$G(0) = H G(-1) + F(0)$$
$$G(1) = H^2 G(-1) + H F(0) + F(1)$$
$$G(2) = H^3 G(-1) + H^2 F(0) + H F(1) + F(2)$$
$$\cdots$$

$$G(n) = \underbrace{H^{n+1} G(-1)}_{\text{free response}} + \underbrace{\sum_{k=0}^{n} H^{n-k} F(k)}_{\text{particular soln.}}$$

Let $HG(-1) = Q \Rightarrow G(n) = H^n Q + \sum_{k=0}^{n} H^{n-k} F(k)$

~~Define~~ Now Z-transform both sides element by element:

$$G(z) = \sum_{n=0}^{\infty} z^n G(n) = H(z)\, Q + H(z)\, F(z) = \left.\begin{matrix} g(z) \\ r(z) \end{matrix}\right]$$

where $H(z) = \left(\sum_{n=0}^{\infty} z^n h_{ij}(n)\right)$, $H^n = \left(h_{ij}(n)\right)$

$$F(z) = \sum_{k=0}^{\infty} F(k) z^k = \begin{pmatrix} \sum f(n) z^n \\ 0 \\ 0 \end{pmatrix}$$

$$H(z) = I + zH + z^2H^2 + \cdots$$

$$(I - zH)\,H(z) = I \quad \Rightarrow \quad \underline{H(z) = (I - zH)^{-1}}$$

Example:

$$g(n) = \tfrac{1}{2}\,g(n-1) + \tfrac{1}{2}\,r(n-1) + \left(\tfrac{1}{2}\right)^n$$
$$r(n) = \tfrac{1}{4}\,g(n-1) + \tfrac{3}{4}\,r(n-1)$$

$$H = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix} \Rightarrow H(z) = \begin{bmatrix} 1-\frac{1}{2}z & -\frac{1}{2}z \\ -\frac{1}{4}z & 1-\frac{3}{4}z \end{bmatrix}^{-1}$$

$$H(z) = \frac{\begin{bmatrix} 1-\frac{3}{4}z & \frac{1}{2}z \\ \frac{1}{4}z & 1-\frac{1}{2}z \end{bmatrix}}{(1-\frac{1}{4}z)(1-z)}$$

$$G(z) = \frac{\begin{bmatrix} 1-\frac{3}{4}z & \frac{1}{2}z \\ \frac{1}{4}z & 1-\frac{1}{2}z \end{bmatrix}\begin{bmatrix} g_1 + \frac{1}{1-\frac{1}{2}z} \\ g_2 \end{bmatrix}}{(1-\frac{1}{4}z)(1-z)} = \begin{bmatrix} g(z) \\ r(z) \end{bmatrix}$$

Note that the eigenvalues are $(\frac{1}{4}, 1)$.

Matrix equations and flow graphs:

Consider the equation

$$X_{n+1}] = H\,X_n] + F_{n+1}]$$

which becomes, when z-transformed:

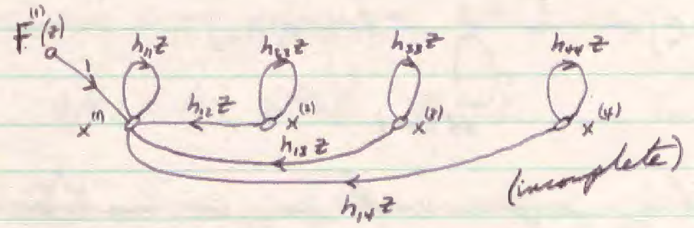
$$X(z) = H(z)\,Q + H(z)\,F(z)$$

where $\quad H(z) = (I - zH)^{-1}$

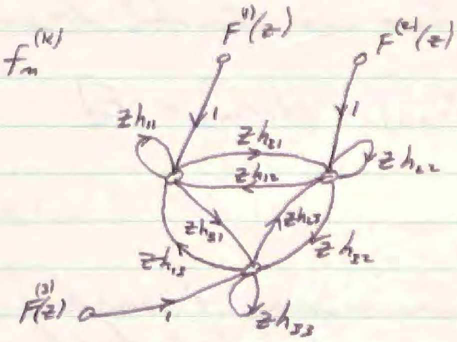
$$Q = H X_{-1}]$$

Then we can draw a flow graph ~~of~~ with nodes $x^{(1)}, x^{(2)}, \ldots,$ from the set of equations

$$x_n^{(1)} = h_{11} x_{n-1}^{(1)} + h_{12} x_{n-1}^{(2)} + \cdots + h_{1K} x_{n-1}^{(K)} + f_n^{(1)}$$
$$x_n^{(2)} = h_{21} x_{n-1}^{(1)} + h_{22} x_{n-1}^{(2)} + \cdots$$
$$x_n^{(K)} = h_{K1} x_{n-1}^{(1)} \cdots + h_{KK} x_{n-1}^{(K)} + f_n^{(K)}$$

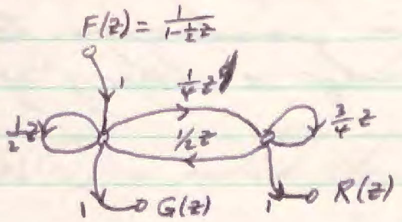


or



Example:

$$g(n) = \tfrac{1}{2} g(n-1) + \tfrac{1}{2} r(n-1) + \left(\tfrac{1}{2}\right)^n$$
$$r(n) = \tfrac{1}{4} g(n-1) + \tfrac{3}{4} r(n-1)$$



$$G(z) = \left(\frac{1}{1-\frac{1}{2}z}\right) \frac{1-\frac{3}{4}z}{1-\frac{1}{2}z-\frac{3}{4}z-\frac{1}{8}z^2+\frac{3}{8}z^2}$$

$$1-\tfrac{1}{2}z-\tfrac{3}{4}z-\tfrac{1}{8}z^2+\tfrac{3}{8}z^2 = \Delta(z)$$

$$X_0] = HX_{-1}] + F_0 = Q + F_0$$

$$Q = H \begin{bmatrix} g(-1) \\ r(-1) \end{bmatrix} = \begin{bmatrix} q^{(1)} \\ q^{(2)} \end{bmatrix}, \quad F_0 = \begin{bmatrix} f_0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

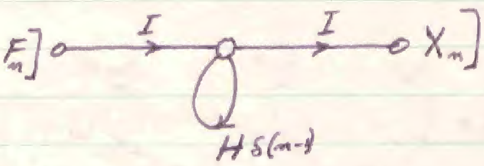
$$G(z) = G_p(z) + \frac{q^{(1)}}{\Delta(z)} + \frac{\frac{1}{2} z\, q^{(2)}}{\Delta(z)}$$

$$H(z) = \begin{bmatrix} \frac{1-\frac{3}{4}z}{\Delta(z)} & \frac{\frac{1}{2}z}{\Delta(z)} \\ \frac{\frac{1}{4}z}{\Delta(z)} & \frac{1-\frac{1}{2}z}{\Delta(z)} \end{bmatrix}$$

Matrix flow graph:

Rather than treat each row of our matrix equation separately, we can consider the signal at each node of a flow graph as a vector having several components. Our matrix equation can then be drawn more simply as:

$$X_{m+1}] + H X_m] + F_{m+1}]$$



The transmission of each branch is expressed as a matrix. The transmission of the flow graph is

$$H(z) = (I - zH)^{-1},$$

corresponding to $\frac{1}{1-zH}$ for scalar flow graphs.

~~[illegible]~~ Proceeding along ~~[illegible]~~ lines analogous to scalar flow graph theory, we can develop a theory of vector flow graphs.

Differential equations:

A general differential equation can be written as

$$\frac{d^k g(t)}{dt^k} = F\left\{ \frac{d^{k-1} g(t)}{dt^{k-1}}, \ldots, g(t), f(t) \right\}$$

If we are given the initial conditions $g(t_0), g'(t_0), \ldots, g^{(k)}(t_0)$, and $f(t)$, we can ~~the thing~~ theoretically find $g(t)$ for all time $t > t_0$.

Suppose that ~~for~~ in the interval $t_m \le t \le t_{m+1}$, $f(t)$ can be characterized by a finite set of numbers

$$M_m].$$

We can also express our initial conditions as a vector

$$G(t_n)\Big] = \left.\begin{matrix} g^{(k-1)}(t)\big|_{t=t_n} \\ g^{(k-2)}(t)\big|_{t=t_n} \\ \vdots \\ g(t)\big|_{t=t_n} \end{matrix}\right]$$

Now we can express $g(t)$ as

$$g(t) = h\{\, G(t_n)],\ M_n] \,\} \quad , \quad t_n \le t \le t_{n+1}.$$

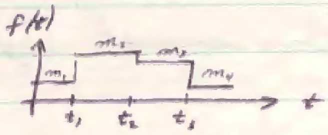
Further, since we know $g(t)$ completely over this region, we can find all the derivatives at all times $t_n \le t \le t_{n+1}$, and in particular, at t_{n+1}. Thus we know $G(t_{n+1})$, so we can further write

$$G(t_{n+1})] = H\{\, G(t_n)],\ M_n] \,\} \quad \text{for all } n,$$

where H is now a matrix operator.

$G(t_n)]$ is called the state of the system at time t_n.

Note that if $t_{n+1} - t_n \to 0$, we can write a first order differential matrix equation which will represent a non-linear system.

<u>Example</u>: $\dfrac{d^2 g(t)}{dt^2} + g(t) = f(t)$; 

; $G(t_0)] = \left.\begin{matrix} g'(0) \\ g(0) \end{matrix}\right] = \left.\begin{matrix} g^{(1)} \\ g^{(2)} \end{matrix}\right]$

$$G(t_n)] = \left.\begin{matrix} g'(n) \\ g(n) \end{matrix}\right] \ ; \ M_n] = m_n$$

$$g(t) = g^{(1)} \sin t + g^{(2)} \cos t + m_0 (1 - \cos t) \quad , \quad 0 < t < 1$$
$$g'(t) = g^{(1)} \cos t - g^{(2)} \sin t + m_0 \sin t \quad , \quad 0 < t < 1$$

$$\left.\begin{matrix} g'(1) \\ g(1) \end{matrix}\right] = \begin{bmatrix} \cos(1) & -\sin(1) \\ \sin(1) & \cos(1) \end{bmatrix} \begin{bmatrix} g'(0) \\ g(0) \end{bmatrix} + m_0 \begin{bmatrix} \sin(1) \\ \cos(1) \end{bmatrix}$$

or $$G_{n+1}] = \begin{bmatrix} \cos T & -\sin T \\ \sin T & \cos T \end{bmatrix} G_n] + m_n \left.\begin{matrix} \sin T \\ \cos T \end{matrix}\right]$$

Discrete systems and random processes:

To specify a discrete random process in general would require infinitely many distributions:

$$P\{f(n)\}, \quad P\{f(n), f(m)\}, \quad \ldots$$

We will restrict our attention to stationary random processes. We can calculate the ensemble average $\overline{f(n)}$ and the time average $\langle f(n) \rangle$. For ergodic processes, $\langle f(n) \rangle = \overline{f(n)}$.

Correlation functions:

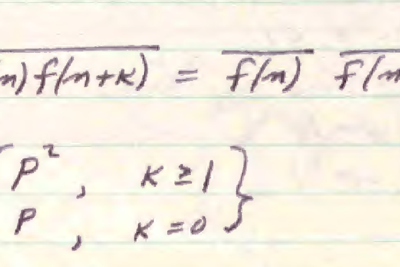
The *autocorrelation function* of a random discrete-time signal is defined as

$$\boxed{\varphi_{ff}(k) \equiv \overline{f(n)f(n+k)}}$$

The *cross-correlation function* of two random discrete-time signals is similarly defined as

$$\boxed{\varphi_{fg}(k) \equiv \overline{f(n)g(n+k)}}$$

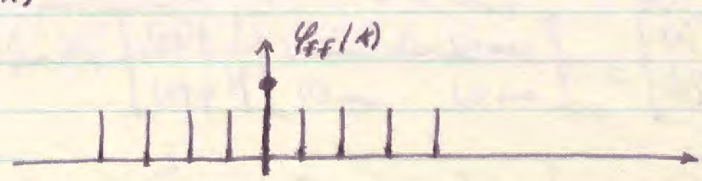
Example: Coin-toss signal input:



successive samples independent

$$\varphi_{ff}(k) = \overline{f(n)f(n+k)} = \overline{f(n)}\;\overline{f(n+k)}$$

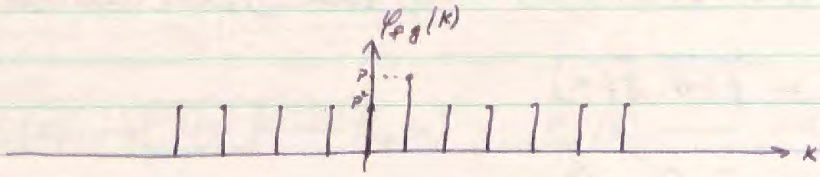
$$= \begin{Bmatrix} p^2, & k \geq 1 \\ p, & k = 0 \end{Bmatrix}$$



Suppose we now put this signal into a unit-delay system:

$$f(n) \longrightarrow \boxed{\delta(n-1)} \longrightarrow g(n) = f(n-1)$$

$$\varphi_{fg}(k) = \overline{f(n)\,g(n+k)} = \overline{f(n)\,f(n+k-1)} = \varphi_{ff}(k-1)$$



$$\varphi_{gg}(k) = \overline{g(n)\,g(n+k)} = \overline{f(n-1)\,f(n+k-1)} = \overline{f(m)\,f(m+k)} = \varphi_{ff}(k)$$

<u>Some properties of autocorrelation functions</u>:

$$0 \le \overline{[f(n) \pm f(n+k)]^2} = \overline{f^2(n)} \pm 2\,\overline{f(n)\,f(n+k)} + \overline{f^2(n+k)}$$

$$0 \le \varphi_{ff}(0) \pm 2\varphi_{ff}(k) + \varphi_{ff}(0) = 2\left[\varphi_{ff}(0) \pm \varphi_{ff}(k)\right]$$

$$0 \le \varphi_{ff}(0) \pm \varphi_{ff}(k) = \varphi_{ff}(0) \pm |\varphi_{ff}(k)| \Rightarrow \varphi_{ff}(0) - |\varphi_{ff}(k)| \ge 0$$

$$\Rightarrow \boxed{\varphi_{ff}(0) \ge |\varphi_{ff}(k)|}$$

$$\boxed{\varphi_{ff}(0) = \overline{f(n)\,f(n)} = \overline{f^2(n)} \ge 0.}$$

Let $s(n) = f(n) - c \Rightarrow \varphi_{ss}(k) = \overline{[f(n)-c][f(n+k)-c]} = \varphi_{ff}(k) - 2c\,\overline{f(n)} + c^2$

Now as k gets large, $f(n+k)$ becomes less dependent on $f(n)$. So we intuitively expect

$$\boxed{\varphi_{ff}(k) \to \overline{f(n)}^{\,2} \text{ for large } k}$$

Now, for large k, $\varphi_{ss}(k) \to \overline{f(n)}^{\,2} - 2c\,\overline{f(n)} + c^2 = \left[\overline{f(n)} - c\right]^2$

Thus we can always make $\varphi_{ss}(k) \to 0$ for large k by subtracting a constant from the signal equal to the mean of $f(n)$. Then we can restrict ourselves to signals of zero mean: $\boxed{\begin{array}{l} s(n) = f(n) - \overline{f(n)} \\ \varphi_{ss}(k) \to 0 \text{ for large } k \end{array}}$

If $\underline{s(n) = r(n) + f(n)}$, then

$$\underline{\varphi_{ss}(k) = \varphi_{rr}(k) + \varphi_{rf}(k) + \varphi_{fr}(k) + \varphi_{ff}(k)}$$

If $f(n)$ and $r(n)$ are either (1) uncorrelated, or (2) are statistically independent and either $\overline{r(n)} = 0$ or $\overline{f(n)} = 0$, then $\varphi_{ss}(k) = \varphi_{ff}(k) + \varphi_{rr}(k)$.

$$\begin{cases} \text{Independence}: & \overline{f(n)\,g(m)} = \overline{f(n)}\;\overline{g(m)} \\ \text{Uncorrelated}: & \overline{f(n)\,g(n+k)} \equiv 0, \text{ all } k \end{cases}$$

Correlation functions and system response:

$$f(n) \longrightarrow \boxed{h(n)} \longrightarrow g(n)$$

Given $\varphi_{ff}(k)$, find $\varphi_{fg}(k)$ and $\varphi_{gg}(k)$:

$$\varphi_{fg}(k) = \overline{f(n)\,g(n+k)} = \overline{f(n)\sum_{m=-\infty}^{\infty} h(m)\,f(n+k-m)} = \sum_m h(m)\,\overline{f(n)\,f(n+k-m)}$$

$$\boxed{\varphi_{fg}(k) = \sum_m h(m)\,\varphi_{ff}(k-m)} \quad \text{or} \quad \varphi_{fg} = h * \varphi_{ff}$$

$$\boxed{\varphi_{ff}(k) \longrightarrow \boxed{h(k)} \longrightarrow \varphi_{fg}(k)}$$

$$\varphi_{gg}(k) = \overline{g(n)\,g(n+k)} = \overline{\sum_m h(m)\,f(n-m)\sum_i h(i)\,f(n+k-i)}$$

$$= \sum_m \sum_i h(m)\,h(i)\,\overline{f(n-m)\,f(n+k-i)} = \sum_m \sum_i h(m)\,h(i)\,\varphi_{ff}(k-i+m)$$

$$\boxed{\varphi_{gg}(k) = \sum_m \sum_j h(m)\,h(j+m)\,\varphi_{ff}(k-j)} \quad \text{where } j = i - m$$

$$\varphi_{gg}(k) = \sum_j \left\{\sum_m h(m)\,h(m+j)\right\} \varphi_{ff}(k-j) \qquad \boxed{\varphi_{ff}(k) \longrightarrow \boxed{\varphi_{hh}(k)} \longrightarrow \varphi_{gg}(k)}$$

$$\boxed{\varphi_{gg}(k) = \sum_j \varphi_{hh}(j)\,\varphi_{ff}(k-j)} \quad \boxed{\text{where } \varphi_{hh}(k) = \sum_m h(m)\,h(m+k)}$$

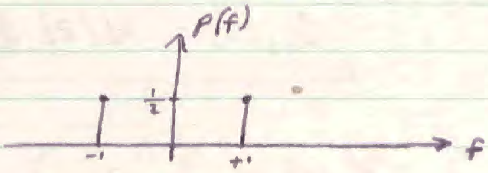
Thus $\boxed{\varphi_{gg} = \varphi_{hh} * \varphi_{ff}}$

$$\varphi_{gg}(k) = \sum_m \sum_i h(m)\,h(i)\,\varphi_{ff}(k-i+m) = \sum_m h(m)\,\varphi_{fg}(k+m) = \sum_m h(m)\,\varphi_{gf}(-k-m)$$

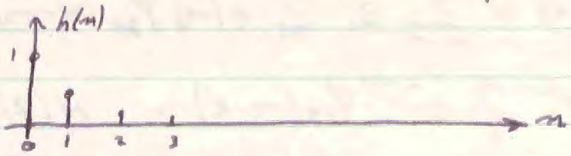
$$\boxed{\varphi_{gg}(k) = \varphi_{gg}(-k) = \sum_m h(m)\,\varphi_{gf}(k-m)}$$

$$\boxed{\varphi_{ff}(k) \longrightarrow \boxed{h(k)} \longrightarrow \varphi_{fg}(k) \quad ; \quad \varphi_{gf}(k) \longrightarrow \boxed{h(k)} \longrightarrow \varphi_{gg}(k)}$$

Example: $f(n) = \begin{Bmatrix} +1, & P(+1) = \frac{1}{2} \\ -1, & P(-1) = \frac{1}{2} \end{Bmatrix}$

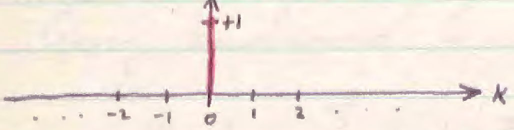


$h(n) = \left(\frac{1}{3}\right)^n, \; n \geq 0$



$$\varphi_{ff}(k) = \overline{f(n)f(n+k)} = \begin{Bmatrix} \overline{f(n)^2} = 1, & k = 0 \\ 0, & k \neq 0 \end{Bmatrix} = \delta(k)$$

$\varphi_{ff}(k) = \delta(k)$



$$\varphi_{fg}(k) = \sum_n h(n)\,\varphi_{ff}(k-n) = \sum_{n=0}^{\infty} \left(\tfrac{1}{3}\right)^n \delta(k-n) = \underbrace{\left(\tfrac{1}{3}\right)^n, \; n \geq 0}, = h(n)$$

$$\varphi_{gg}(k) = \sum_n h(n)\,\varphi_{gf}(k-n) = \sum_{n=k}^{\infty} \left(\tfrac{1}{3}\right)^n \left(\tfrac{1}{3}\right)^{n-k}, \; k \geq 0, \text{ since } \varphi_{gf}(n) = \left(\tfrac{1}{3}\right)^{-n}, \; -n \geq 0$$

$$= \left(\tfrac{1}{3}\right)^{-k} \sum_{n=k}^{\infty} \left(\tfrac{1}{3}\right)^{2n} = \left(\tfrac{1}{3}\right)^{+k} \sum_{m=0}^{\infty} \left(\tfrac{1}{3}\right)^{2m} = \left(\tfrac{1}{3}\right)^{k} \sum_{m=0}^{\infty} \left(\tfrac{1}{9}\right)^{m}, \quad k \geq 0$$

$$= \tfrac{9}{8}\left(\tfrac{1}{3}\right)^{k}, \quad k \geq 0$$

so $\underline{\varphi_{gg}(k) = \frac{9}{8}\left(\frac{1}{3}\right)^{|k|}, \; \text{all } k}$

Transforms of correlation functions:

Let the z-transform (double-ended) of $\varphi_{ff}(z)$ be defined as

$$\boxed{\Phi_{ff}(z) \equiv \sum_{k=-\infty}^{\infty} \varphi_{ff}(k)\, z^{k}}$$

$$\Phi_{fg}(z) = \sum_{k=-\infty}^{\infty} \varphi_{fg}(k)\, z^{k} = \sum_{k=-\infty}^{\infty} z^{k} \sum_{j=-\infty}^{\infty} h(j)\, \varphi_{ff}(k-j) = \sum_{j=-\infty}^{\infty} h(j)\, z^{j} \sum_{k=-\infty}^{\infty} z^{k-j} \varphi_{ff}(k-j)$$

or

$$\boxed{\Phi_{fg}(z) = H(z)\, \Phi_{ff}(z)}$$

$$\Phi_{gg}(z) = \sum_{k=-\infty}^{\infty} z^{k} \varphi_{gg}(k) = \sum_{k=-\infty}^{\infty} z^{k} \sum_{m} h(m)\, \varphi_{gf}(k-m) = \sum_{k} z^{k} \sum_{m} h(m)\, \varphi_{fg}(m-k)$$

$$= \sum_{m} h(m)\, z^{m} \sum_{k} z^{k-m} \varphi_{fg}(m-k) = H(z)\, \Phi_{fg}\left(\tfrac{1}{z}\right) = H(z) H\left(\tfrac{1}{z}\right) \Phi_{ff}\left(\tfrac{1}{z}\right)$$

$$\cancel{\text{or} \;= \sum_{k} z^{k} \sum_{m} h(k-m)\, \varphi_{ff}(-m) = \sum_{m} z^{m} \varphi_{fg}(-m) \sum_{k} z^{k-m} h(k-m)}$$

But $\varphi_{ff}(k) = \varphi_{ff}(-k)$, so $\Phi_{ff}(z) = \sum_{k=-\infty}^{\infty} \varphi_{ff}(-k)\, z^{k} = \sum \varphi_{ff}(k)\, z^{-k} = \Phi_{ff}\left(\tfrac{1}{z}\right)$

$$\boxed{\Phi_{ff}\left(\tfrac{1}{z}\right) = \Phi_{ff}(z)}$$

so

$$\boxed{\Phi_{gg}(z) = H(z) H\left(\tfrac{1}{z}\right) \Phi_{ff}(z)}$$

Note that since $\varphi_{ff}(k)$ is an even function, we can always write $\Phi_{ff}(z) = H(z) H\left(\frac{1}{z}\right)$.

Also, $\Phi_{ff}(z) = \dfrac{P_1(z)\, P_2(z)}{Q(z)\, Q_2(z)}$, P's have all singularities outside $|z| = 1$, Q's " " " inside $|z| = 1$.

Also, we can turn any signal into white noise by passing it through a filter with response $\frac{1}{H(z)}$

$$\Phi_{ff}(z) = H(z) H\left(\tfrac{1}{z}\right) \leftrightarrow f(n) \longrightarrow \boxed{\tfrac{1}{H(z)}} \longrightarrow g(n) \leftrightarrow \Phi_{gg}(z) = \frac{1}{H(z)} \frac{1}{H\left(\frac{1}{z}\right)} \Phi_{ff}(z) = 1$$

Example:

$$\Phi_{ff}(z) = 1 \leftrightarrow \varphi_{ff}(k) = \delta(k)$$

$$H(z) = \frac{1}{1-\frac{1}{3}z} \leftrightarrow h(n) = \left(\tfrac{1}{3}\right)^n, \quad n \geq 0$$

$$\Phi_{fg}(z) = H(z)\,\Phi_{ff}(z) = \frac{1}{1-\frac{1}{3}z}$$

$$\Phi_{gg}(z) = \frac{1}{1-\frac{1}{3}z}\,\frac{1}{1-\frac{1}{3z}} = \frac{1}{1-\frac{1}{3}z}\,\frac{(-3z)}{1-3z}$$

Now $\varphi_{gg}(k)$ must be bounded for all k, so

$$\Phi_{gg}(z) = \underbrace{\frac{9/8}{1-\frac{1}{3}z}}_{k\geq 0} + \underbrace{\frac{-9/8}{1-3z}}_{k\leq 0} = \underbrace{\frac{9/8}{1-\frac{1}{3}z}}_{k\geq 0} + \underbrace{\frac{\frac{9}{8}\left(\frac{1}{3z}\right)}{1-\frac{1}{3z}}}_{k<0}$$

$$\varphi_{gg}(k) = \begin{cases} \frac{9}{8}\left(\frac{1}{3}\right)^{k}, & k \geq 0 \\ \frac{9}{8}\left(\frac{1}{3}\right)^{-k}, & k < 0 \end{cases}$$

$$\varphi_{gg}(k) = \frac{9}{8}\left(\tfrac{1}{3}\right)^{|k|}, \quad \text{all } k$$

Going backward to $\Phi_{gg}(z)$, we see

$$\Phi_{gg}(z) = \sum_{k=-\infty}^{\infty} \varphi_{gg}(k)\, z^k = \frac{9}{8}\sum_{k=0}^{\infty}\left(\tfrac{1}{3}\right)^k z^k + \frac{9}{8}\sum_{k=-\infty}^{-1}\left(\tfrac{1}{3}\right)^{-k} z^k$$

$$= \left(\tfrac{9}{8}\right)\frac{1}{1-\frac{1}{3}z} + \left(\tfrac{9}{8}\right)\sum_{k=1}^{\infty}\left(\tfrac{1}{3}\right)^k\left(\tfrac{1}{z}\right)^k = \frac{9/8}{1-\frac{1}{3}z} + \left(\tfrac{9}{8}\right)\left(\tfrac{1}{3z}\right)\sum_{k=0}^{\infty}\left(\tfrac{1}{3z}\right)^k$$

$$= \frac{9/8}{1-\frac{1}{3}z} + \frac{(9/8)\left(\frac{1}{3z}\right)}{1-3z}$$

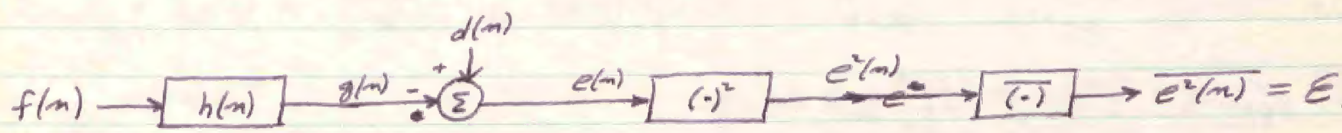
Optimal discrete linear system :

Suppose we have an input function $f(n)$ and we want to put it through some linear system $h(n)$ to get an output $g(n)$ as close as possible to our desired output $d(n)$.

$$f(n) \longrightarrow \boxed{\text{DESIRED SYSTEM}} \longrightarrow d(n) = \text{desired output}$$

$$f(n) \longrightarrow \underset{h(n)}{\boxed{\text{LINEAR SYSTEM}}} \longrightarrow g(n) = \text{actual output}$$

The only error criterion we can handle is mean square error. Thus we will try to select an $h(n)$ which will minimize the mean square error $\overline{e^2(n)}$.



$$\mathcal{E} = \overline{e^2(n)} = \overline{[d(n)-g(n)]^2} = \overline{d^2(n)} - 2\overline{d(n)g(n)} + \overline{g^2(n)}$$

$$= \varphi_{dd}(0) - 2\overline{d(n)\sum_m f(n-m)h(m)} + \overline{\sum_m h(m)f(n-m)\sum_l h(l)f(n-l)}$$

$$= \varphi_{dd}(0) - 2\sum_m h(m)\overline{d(n)f(n-m)} + \sum_m\sum_l h(m)h(l)\overline{f(n-m)f(n-l)}$$

$$= \varphi_{dd}(0) - 2\sum_m h(m)\varphi_{df}(-m) + \sum_m h(m)\sum_l h(l)\varphi_{ff}(m-l)$$

$$\boxed{\mathcal{E} = \varphi_{dd}(0) - 2\sum_m h(m)\varphi_{fd}(m) + \sum_m h(m)\sum_l h(l)\varphi_{ff}(m-l)}$$

We now want to minimize this $\mathcal{E}$ over $h(n)$ for each n:

$$\frac{\partial\mathcal{E}}{\partial h(n)} = 0 - 2\varphi_{fd}(n) + 2\sum_m h(m)\varphi_{ff}(n-m) = 0 \quad \text{for each } n.$$

Thus the optimal system must satisfy the equation

$$\boxed{\sum_{m=-\infty}^{\infty} h_o(m)\varphi_{ff}(n-m) = \varphi_{fd}(n), \quad -\infty < n < \infty}$$

Realizability not considered.

If we now require our system to operate in real time, we must require physical realizability, i.e., $h(n) = 0,\ n < 0$. The only effect this has is to change our summation limits from $(-\infty, \infty)$ to $(0, \infty)$. so

$$\boxed{\sum_{m=0}^{\infty} h_o(m)\,\varphi_{ff}(n-m) = \varphi_{fd}(n) \quad,\ n \geq 0 : \text{optimal realizable system}}$$

<u>Discrete version of Wiener-Hopf equation.</u>

<u>Proof of optimality</u>:

We have rigorously only shown that an $h(n)$ satisfying one of the above equations gives a local max or min of $\mathcal{E}$. Thus we will try a variational technique to show that the system found above is truly optimal.

Let $h(n) = h_o(n) + \eta(n)$, where $h(n)$ satisfies the above optimality equations. The mean square error associated with this system is

$$\mathcal{E}_h = \varphi_{dd}(0) - 2\sum h_o(m)\varphi_{fd}(m) - 2\sum \eta(n)\varphi_{fd}(n)$$

$$+ \sum_m \sum_l h_o(m) h_o(l)\,\varphi_{ff}(m-l) + 2\sum_m\sum_l h_o(m)\,\eta(l)\,\varphi_{ff}(m-l) + \sum_m\sum_l \eta(m)\,\eta(l)\,\varphi_{ff}(m-l)$$

$$\mathcal{E}_h = \varphi_{dd}(0) - \sum_m h_o(m)\,\varphi_{fd}(m) + \sum_m \sum_l \eta(m)\,\eta(l)\,\varphi_{ff}(m-l)$$

Now this last term is always positive and is the only term dependent on $\eta(\cdot)$. $\mathcal{E}_h$ will clearly be an absolute minimum if $\eta(\cdot) \equiv 0$. Thus, $h_o(n)$ as found above <u>is</u> the optimal linear system in the sense that it minimizes the m.s.e. $\mathcal{E}$.

$$\underline{\mathcal{E}_{min} = \varphi_{dd}(0) - \sum_m h_o(m)\,\varphi_{fd}(m)}$$

The optimal system has decreased the error over no system at all by $\sum_m h_o(m)\,\varphi_{fd}(m)$.

Example of finding optimal system:

Let $\varphi_{ff}(k) = \frac{9}{8}\left(\frac{1}{3}\right)^{|k|}$ ~~and~~

$d(n) = f(n+P)$: we desire a prediction.

$\varphi_{fd}(k) = \overline{f(n)\,d(n+k)} = \overline{f(n)\,f(n+k+P)} = \varphi_{ff}(k+P)$

$\varphi_{fd}(k) = \frac{9}{8}\left(\frac{1}{3}\right)^{|k+P|}$

The Weiner-Hopf equation then becomes

$$\sum_{m=-\infty}^{\infty} h_o(m)\,\varphi_{ff}(n-m) = \varphi_{fd}(n)$$

or $$\frac{9}{8}\sum_{m=-\infty}^{\infty} h_o(m)\left(\frac{1}{3}\right)^{|n-m|} = \frac{9}{8}\left(\frac{1}{3}\right)^{|n+P|} \qquad \text{all } n$$

The obvious solution to this equation is:

$$h_o(m) = \delta(m+P)$$

This just says: "If you want to predict, build a predictor." Unfortunately we do not know how to build a predictor, so we will look at the realizable case:

$$\sum_{m=0}^{\infty} h_o(m)\left(\frac{1}{3}\right)^{|n-m|} = \left(\frac{1}{3}\right)^{|n+P|} \quad , \; n \geq 0$$

or $$\sum_{m=0}^{\infty} h_o(m)\left(\frac{1}{3}\right)^{n-m} = \left(\frac{1}{3}\right)^{n+P} \quad , \; n \geq 0$$

The solution to this equation is:

$$h_o(m) = \left(\frac{1}{3}\right)^{P}\delta(m) \quad ; \text{ realizable}$$

[$\left(\frac{1}{3}\right)^{P+K}\delta(m-K)$ is also a solution if $K \geq 0$]

Here, $E_{min} = \varphi_{dd}(0) - \sum h_o(m)\,\varphi_{fd}(m) = \frac{9}{8} - \frac{9}{8}\sum_{m=0}^{\infty}\left(\frac{1}{3}\right)^{P}\delta(m)\left(\frac{1}{3}\right)^{m+P}$

$$= \frac{9}{8}\left[1 - \left(\frac{1}{3}\right)^{2P}\right] = \frac{9}{8}\left[1 - \left(\frac{1}{9}\right)^{P}\right]$$

Finding optimum (NNPR) systems in the transform domain:

The Wiener-Hopf equation for a "not-necessarily-physically-realizable" [NNPR] system is

$$\sum_{i=-\infty}^{\infty} h_o(i)\,\varphi_{ff}(n-i) = \varphi_{fd}(n) \quad , -\infty < n < \infty$$

If we z-transform this equation, we get

$$H_o(z)\Phi_{ff}(z) = \Phi_{fd}(z)$$

So the optimal NNPR system is

$$\boxed{H_o(z) = \frac{\Phi_{fd}(z)}{\Phi_{ff}(z)}}$$

Example: $\varphi_{ff}(k) = \frac{9}{8}\left(\frac{1}{3}\right)^{|k|}, -\infty < k < \infty$; $d(n) = f(n+P)$: pure prediction.

$$\varphi_{fd}(k) = \overline{f(n)d(n+k)} = \overline{f(n)f(n+P+k)} = \varphi_{ff}(k+P)$$

$$\Phi_{fd}(z) = \sum z^k \varphi_{ff}(k+P) = z^{-P}\,\Phi_{ff}(z)$$

The optimal NNPR system is then

$$H_o(z) = \frac{z^{-P}\Phi_{ff}(z)}{\Phi_{ff}(z)} = z^{-P} \longleftrightarrow \underline{h_o(n) = \delta(n+P)}$$

A predictor is clearly the best predictor, but is not realizable.

Reproduction of signal in noise:

We can formulate the problem of filtering a desired signal from noise in the same form as that just studied.

Let $f(n) = d(n) + r(n)$, d and r are uncorrelated.

$$\varphi_{ff}(k) = \varphi_{dd}(k) + \varphi_{rr}(k) \leftrightarrow \Phi_{ff}(z) = \Phi_{dd}(z) + \Phi_{rr}(z)$$

$$\varphi_{fd}(k) = \overline{f(n)d(n+k)} = \overline{d(n)d(n+k)} + \overset{\nearrow 0}{\overline{r(n)\cancel{d(n+k)}}} = \varphi_{dd}(k)$$

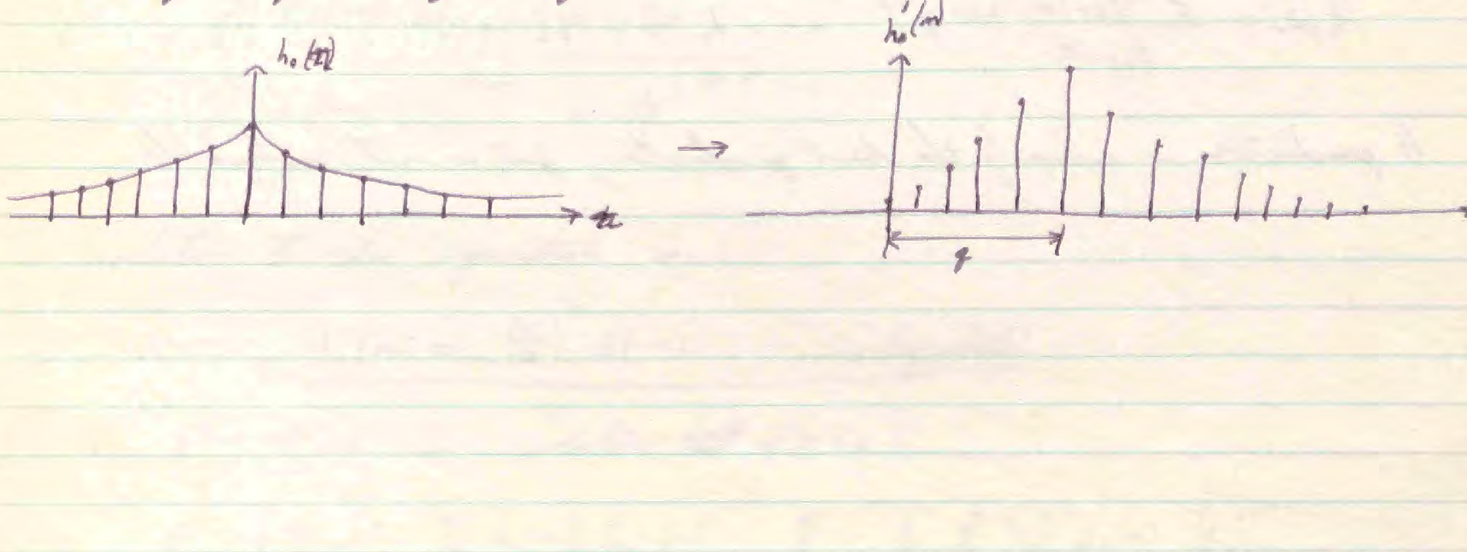
so $\Phi_{fd}(z) = \Phi_{dd}(z)$

Thus, the optimal linear system is

$$\boxed{H_o(z) = \frac{\Phi_{fd}(z)}{\Phi_{ff}(z)} = \frac{\Phi_{dd}(z)}{\Phi_{dd}(z) + \Phi_{rr}(z)}}$$

NNPR

~~Example~~ We can still use this result even if ~~it~~ the system it represents is not physically realizable if we do not mind a delay in getting our information. We can do this by recording the signal for a long time & then analyze the record with a computation not requiring realizability.



Example: $\varphi_{dd}(k) = \frac{10}{27}\left(\frac{1}{2}\right)^{|k|}, \ -\infty < k < \infty$

$\varphi_{rr}(k) = \frac{2}{3}\delta(k)$

} uncorrelated.

$$\Phi_{dd}(z) = \frac{10}{27}\left[\frac{1}{1-\frac{1}{2}z} + \frac{\frac{1}{2z}}{1-\frac{1}{2z}}\right] = \frac{5/18}{(1-\frac{1}{2}z)(1-\frac{1}{2z})}$$

$$\Phi_{rr}(z) = \frac{2}{3}$$

$$\Phi_{dd}(z) + \Phi_{rr}(z) = \frac{\frac{5}{18} + \frac{2}{3}(1-\frac{1}{2}z)(1-\frac{1}{2z})}{(1-\frac{1}{2}z)(1-\frac{1}{2z})} = \frac{(1-\frac{1}{3}z)(1-\frac{1}{3z})}{(1-\frac{1}{2}z)(1-\frac{1}{2z})}$$

So: $$\underbrace{H_0(z) = \frac{\Phi_{dd}(z)}{\Phi_{dd}(z)+\Phi_{rr}(z)} = \frac{5/18}{(1-\frac{1}{3}z)(1-\frac{1}{3z})}}$$ ~~$h_0(m) = 5$~~

$$= \frac{5/16}{1-\frac{1}{3}z} + \frac{\frac{5}{16}\left(\frac{1}{3z}\right)}{1-\frac{1}{3z}} \longleftrightarrow \underline{h_0(m) = \frac{5}{16}\left(\frac{1}{3}\right)^{|m|}}$$

$$\overline{e^2(m)} = \varphi_{dd}(0) - \sum_m h_0(m)\varphi_{fd}(m) = \frac{10}{27} - \sum_m \left(\frac{5}{16}\right)\left(\frac{1}{3}\right)^{|m|}\left(\frac{10}{27}\right)\left(\frac{1}{2}\right)^{|m|}$$

$$= \frac{10}{27}\left[1 - \frac{5}{16}\sum_{-\infty}^{\infty}\left(\frac{1}{6}\right)^{|m|}\right] = \frac{10}{27}\left[1 - \frac{5}{16} - \frac{5}{8}\left(\frac{1}{5}\right)\right] = \frac{10}{27}\left[\frac{9}{16}\right] = \frac{5}{24}$$

System	error
$h(m) = 0$	$\frac{10}{27} = 0.370$
$h_0(m)$ NNPR	$\frac{5}{24} = 0.208$
$h(m) = \delta(m)$	$\frac{2}{3} = 0.667$

see p. for realizable system errors.

Optimal realizable systems :

Let us split our system into two parts :

uncorrelated

$$f(n) \longrightarrow \boxed{\frac{1}{\Phi_{ff}^{+}(z)}} \xrightarrow{f'(n)} \boxed{H_o'(z)} \longrightarrow g(n) \quad \leftrightarrow \quad \boxed{f(n) \longrightarrow \boxed{H_o(z)} \longrightarrow g(n)} \longrightarrow$$

$$\Phi_{ff}(z) = \Phi_{ff}^{+}(z)\,\Phi_{ff}^{-}(z) \quad ; \quad \Phi_{ff}^{+}(z) \text{ has all singularities outside } |z| = 1.$$

$$\Phi_{f'f'}(z) = \frac{1}{\Phi_{ff}^{+}(z)}\,\frac{1}{\Phi_{ff}^{+}(\frac{1}{z})}\,\Phi_{ff}(z) = \frac{\Phi_{ff}(z)}{\Phi_{ff}^{+}(z)\,\Phi_{ff}^{-}(z)} = 1.$$

Now, our equations for the optimal linear systems are

$$\sum_{i=-\infty}^{\infty} h_o(i)\,\varphi_{ff}(n-i) = \varphi_{fd}(n) \quad , -\infty < n < \infty , \quad \underline{NNPR}$$

$$\text{or} \quad \sum_{i=0}^{\infty} h_o(i)\,\varphi_{ff}(n-i) = \varphi_{fd}(n) \quad , \quad n \geq 0 \quad , \quad \underline{PR}$$

We are interested in choosing $H_o'(z) \leftrightarrow h_o'(n)$, and we know that $\Phi_{f'f'}(z) = 1 \leftrightarrow \varphi_{f'f'}(n) = \delta(n)$. The above equations then become

$$h_o'(n) = \varphi_{f'd}(n) \quad , -\infty < n < \infty , \quad \underline{NNPR}$$

$$\text{or} \quad h_o'(n) = \begin{cases} \varphi_{f'd}(n) , & n \geq 0 \\ 0 , & n < 0 \end{cases} \quad , \quad \underline{PR}$$

These two equations are identical for $n \geq 0$. Thus, we get the optimal realizable system by throwing away the negative time samples of the non-realizable system.

We can now find the ~~non~~ realizable optimal system by finding the optimal non-realizable system and throwing away all negative time samples since we know that $\frac{1}{\Phi_{ff}^{+}(z)}$ is realizable. Thus all negative time samples are due to $H_o'(z)$.

$$H_o(z) = \frac{\Phi_{fd}(z)}{\Phi_{ff}(z)} = \frac{1}{\Phi_{ff}^+(z)}\left[\frac{\Phi_{fd}(z)}{\Phi_{ff}^-(z)}\right] \quad NNPR$$

To get the realizable system, we must throw away the negative time samples of $\Phi_{fd}(z)/\Phi_{ff}^-(z)$. That is

$$\boxed{H_o(z) = \frac{1}{\Phi_{ff}^+(z)}\left[\frac{\Phi_{fd}(z)}{\Phi_{ff}^-(z)}\right]_+ \qquad \text{PR optimal system}}$$

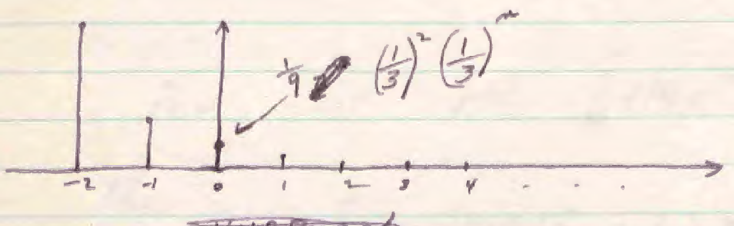
<u>Example</u>: Prediction

$$d(n) = f(n+P), \qquad \varphi_{ff}(k) = \frac{9}{8}\left(\frac{1}{3}\right)^{|k|} \text{ for all } k \ (-\infty, \infty)$$

$$\Phi_{ff}(z) = \frac{1}{(1-\frac{1}{3}z)(1-\frac{1}{3z})} \Rightarrow \Phi_{ff}^+(z) = \frac{1}{(1-\frac{1}{3}z)}$$

$$\Phi_{fd}(z) = z^{-P}\,\Phi_{ff}(z)$$

$$\left[\frac{\Phi_{fd}(z)}{\Phi_{ff}^-(z)}\right]_+ = \left[z^{-P}\,\Phi_{ff}^+(z)\right]_+ = \left[\frac{z^{-P}}{1-\frac{1}{3}z}\right]_+ = \frac{\left(\frac{1}{3}\right)^P}{1-\frac{1}{3}z}$$



~~NNPR syst~~

$$H_o(z) = \frac{1}{\Phi_{ff}^+}\left[\frac{\left(\frac{1}{3}\right)^P}{1-\frac{1}{3}z}\right] = \frac{(1-\frac{1}{3}z)\left(\frac{1}{3}\right)^P}{(1-\frac{1}{3}z)} = \frac{1}{3}P$$

<u>If we ignore realizability,</u>

$$H_o(z) = (1-\tfrac{1}{3}z)\frac{z^{-P}}{1-\frac{1}{3}z} = z^{-P} \quad \checkmark$$

Example: filtering:

$f(n) = r(n) + d(n)$, $\varphi_{dd}(k) = \frac{10}{27}\left(\frac{1}{2}\right)^{|k|}$, all $k(-\infty,\infty)$; $\varphi_{rr}(k) = \frac{2}{3}\,\delta(k)$

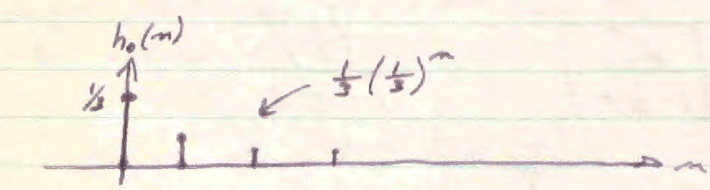
$$\Phi_{fd}(z) = \frac{\frac{5}{18}}{(1-\frac{1}{2}z)(1-\frac{1}{2z})}$$

$$\Phi_{ff}(z) = \frac{(1-\frac{1}{3}z)(1-\frac{1}{3z})}{(1-\frac{1}{2}z)(1-\frac{1}{2z})}$$

$$\frac{\Phi_{fd}(z)}{\Phi_{ff}^{-}(z)} = \frac{5/18}{(1-\frac{1}{2}z)(1-\frac{1}{3z})} = \frac{\frac{1}{3}}{1-\frac{1}{2}z} + \frac{\left(\frac{1}{3}\right)\frac{1}{3z}}{1-\frac{1}{3z}}$$

$$\left[\frac{\Phi_{fd}(z)}{\Phi_{ff}^{-}(z)}\right]_+ = \frac{\frac{1}{3}}{1-\frac{1}{2}z}$$

$$H_0(z) = \frac{1-\frac{1}{2}z}{1-\frac{1}{3}z}\,\frac{\frac{1}{3}}{1-\frac{1}{2}z} = \frac{\frac{1}{3}}{1-\frac{1}{3}z}$$



The error for the realizable system is

$$\overline{e^2(n)} = \frac{10}{27} - \sum_0^\infty \left(\frac{1}{3}\right)^{m+1} \frac{10}{27}\left(\frac{1}{2}\right)^m = \frac{10}{27}\left[1 - \frac{1}{3}\sum_0^\infty \left(\frac{1}{6}\right)^m\right] = \frac{2}{9} = .222$$

	system	error
	$h(n) = 0$	.370
PR	$h(n) = h_0(n)$	.222
NPR	$h(n) = h_0(n)$	.208
	$h(n) = \delta(n)$	.667

} Is the error reduction worth the delay needed to approximate the non-realizable soln?

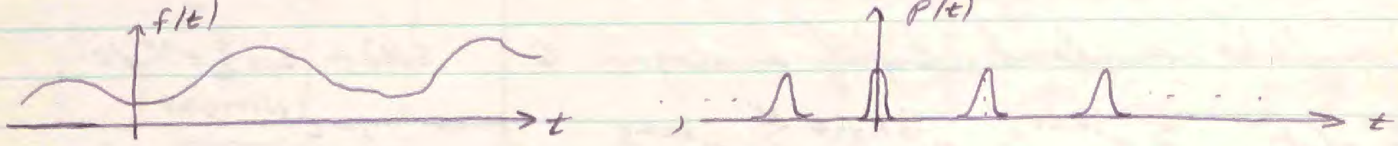
Transforms: It can be shown that as far as properties of integration are concerned, that

$$\frac{1}{2\pi j}\int_{-\infty}^{\infty} e^{st}\,ds = \lim_{N\to\infty} \frac{\sin Nt}{\pi t} \to \delta(t)$$

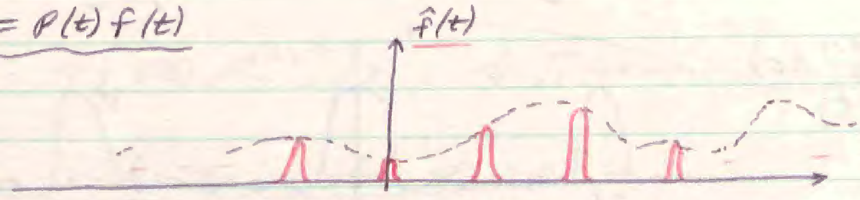
This is the result needed to establish the validity of the exponential transform pair.

Sampling:

Suppose we have a waveform $f(t)$ which we want to sample with some pulse train $p(t)$:



Let $\hat{f}(t) = p(t)\,f(t)$



Now, let $\quad F(s) = \int_{-\infty}^{\infty} f(t)e^{-st}\,dt \quad , \quad P(s) = \int_{-\infty}^{\infty} p(t)e^{-st}\,dt$

Then

$$\hat{F}(s) = \int_{-\infty}^{\infty} f(t)p(t)e^{-st}\,dt = \int dt\, e^{-st}\,\frac{1}{2\pi j}\int d\mathcal{J}\, P(\mathcal{J})e^{\mathcal{J}t}\,\frac{1}{2\pi j}\int d\lambda\, F(\lambda)e^{\lambda t}$$

$$= \frac{1}{2\pi j}\int d\mathcal{J}\, P(\mathcal{J})\int d\lambda\, F(\lambda)\left[\frac{1}{2\pi j}\int dt\, e^{\mathcal{J}t}e^{\lambda t}e^{-st}\right]$$

$$= \frac{1}{2\pi j}\int d\mathcal{J}\, P(\mathcal{J})\int d\lambda\, F(\lambda)\,\delta(\mathcal{J}+\lambda-s)$$

or

$$\boxed{\hat{F}(s) = \frac{1}{2\pi j}\int d\mathcal{J}\, P(\mathcal{J})\,F(s-\mathcal{J}) = \frac{1}{2\pi j}\int d\lambda\, P(s-\lambda)\,F(\lambda)}$$

Impulse trains and their transforms:

$$p(t) = \sum_{-\infty}^{\infty} {}_n\, \delta(t-nT)$$

The impulse train

can be represented as the Fourier series

$$p(t) = \sum_{-\infty}^{\infty} {}_k\, C_k e^{+jk\Omega t} \quad , \quad \Omega \equiv \frac{2\pi}{T}$$

where $C_k = \frac{1}{T}\int_{-T/2}^{T/2} p(t)\, e^{-jk\Omega t}\, dt$

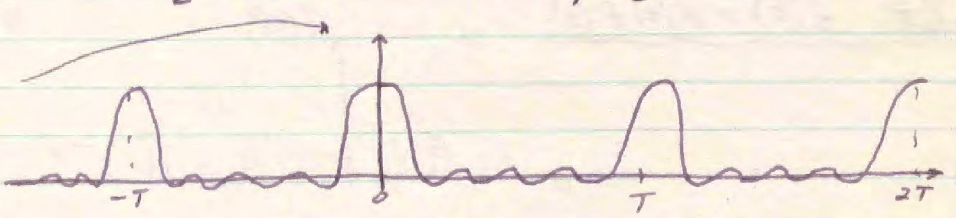
$$= \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)\, e^{-jk\Omega t} = \frac{1}{T}$$

Thus, $\boxed{p(t) = \sum_{-\infty}^{\infty} {}_k\, \delta(t-kT) = \frac{1}{T}\sum_{-\infty}^{\infty} {}_k\, e^{jk\Omega t}}$ Fourier series

This might be considered slightly analogous to $\delta(t) = \frac{1}{2\pi j}\int_{-\infty}^{\infty} e^{st}\, ds$.

Note that $\sum_{-N}^{N} {}_k\, e^{jk\Omega t} = e^{-jN\Omega t}\sum_{0}^{2N} {}_k\, e^{jk\Omega t} = e^{jN\Omega t}\,\frac{1-e^{j(2N+1)\Omega t}}{1-e^{j\Omega t}}$

$$= \frac{\sin\left(\frac{2N+1}{2}\Omega t\right)}{\sin\left(\frac{\Omega t}{2}\right)}$$



As $N \to \infty$, this approaches an impulse train

Now, the exponential transform of $p(t)$ can be written:

$$P(s) = \int_{-\infty}^{\infty} p(t)\, e^{-st}\, dt = \sum_{-\infty}^{\infty} {}_k \int_{(k-\frac{1}{2})T}^{(k+\frac{1}{2})T} p(t)\, e^{-st}\, dt$$

But $p(t)$ has period T, so

$\boxed{P(s) = \sum_{-\infty}^{\infty} {}_k\, e^{-skT}\int_{-T/2}^{T/2} p(t)\, e^{-st}\, dt = \Pi(s)\sum_{-\infty}^{\infty} {}_k\, e^{-skT}}$ exponential transform

where $\Pi(s)$ is the transform of one pulse of the pulse train.

Recall from the last page that

$$\sum \delta\left(x - \frac{2\pi k}{a}\right) = \frac{a}{2\pi} \sum e^{jkax}$$

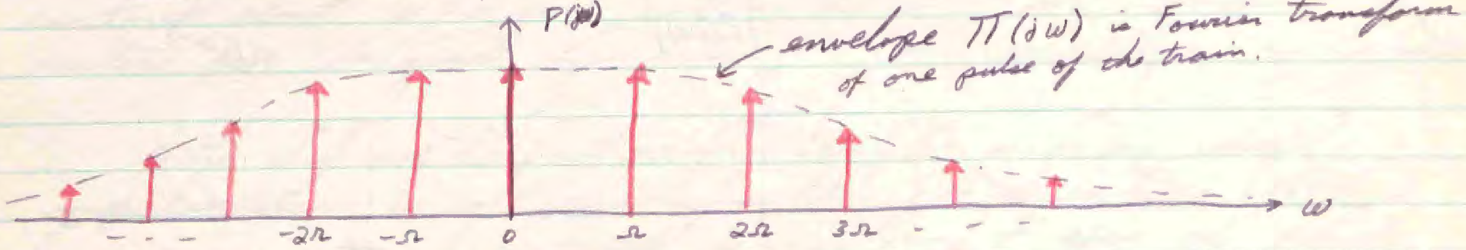
If we let $a = -jT$, $x = -s$, then we get

$$\sum e^{-skT} = \frac{2\pi j}{T} \sum \delta\left(s + j\frac{2\pi k}{T}\right) = j\Omega \sum \delta(s + jk\Omega)$$

Thusly,

$$\boxed{P(s) = j\Omega\, \Pi(s) \sum_{-\infty}^{\infty} \delta(s + jk\Omega)}$$ for pulse train

or $$\boxed{P(s) = j\Omega \sum_{-\infty}^{\infty} \Pi(-jk\Omega)\, \delta(s + jk\Omega)}$$



Note: $c_k = \frac{1}{T}\int_{-T/2}^{T/2} p(t)\, e^{-jk\Omega t}\, dt = \frac{1}{T}\Pi(jk\Omega)$

So the area under the impulse at $k\Omega$ is:

$$j\Omega\, \Pi(jk\Omega) = \Omega T c_k = 2\pi c_k$$

Inverse transform:

$$\frac{1}{2\pi j}\int P(s)\, e^{-st}\, ds = \frac{1}{T} \sum \delta(t - kj\Omega)\, \Pi(s)\, e^{st}\, ds$$

$$= \sum \underbrace{\frac{\Pi(jk\Omega)}{T}}_{c_k} e^{jk\Omega t} = \sum c_k e^{jk\Omega t} = p(t) \checkmark$$

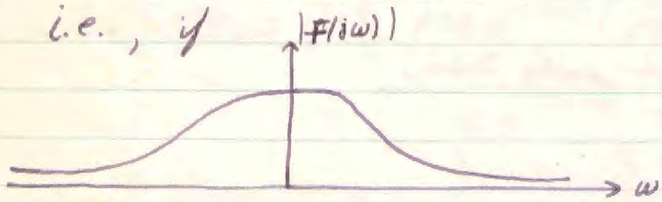
We can now write the transform of the sampled waveform as

$$\hat{F}(s) = \frac{1}{2\pi j}\int F(\lambda)P(s-\lambda)\,d\lambda = \frac{1}{T}\int_{-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\delta(s-\lambda-jk\Omega)\,\Pi(s-\lambda)\,F(\lambda)\,d\lambda$$

$$= \frac{1}{T}\sum_{k=-\infty}^{\infty} F(s+jk\Omega)\,\Pi(-jk\Omega)$$

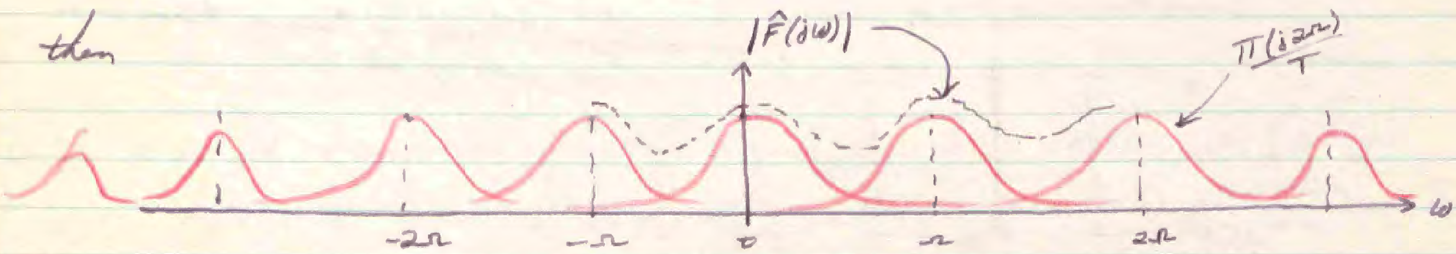
$$\boxed{\hat{F}(s) = \frac{1}{T}\sum_{k=-\infty}^{\infty} F(s-jk\Omega)\,\Pi(jk\Omega)}$$ general pulse train sampling

i.e., if



~~then~~

then



If we use an impulse train for our sampling, we have

$$\Pi(s) = 1$$

so $\quad F^*(s) = \frac{1}{T}\sum F(s+jk\Omega)$

or $\quad \boxed{F^*(s) = \frac{1}{T}\sum_{k=-\infty}^{\infty} F(s-jk\Omega)}$ impulse train sampling.

We can also write:

$z = e^{-sT}$

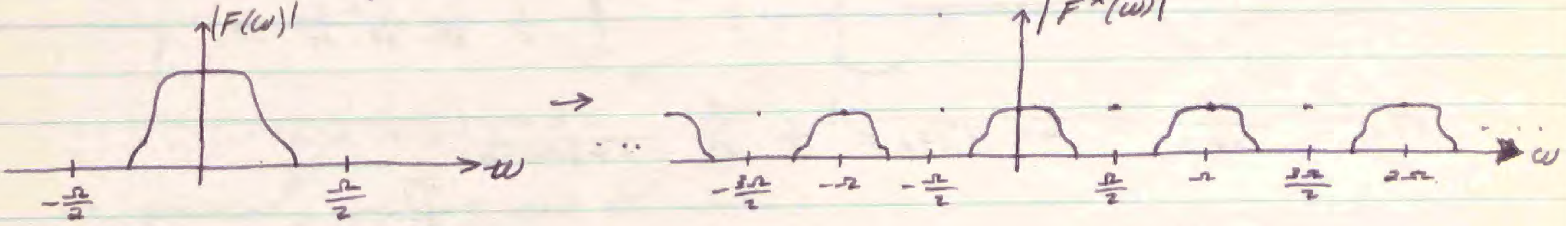
$$F^*(s) = \int_{-\infty}^{\infty} f(t)\sum_{n=-\infty}^{\infty}\delta(t-nT)\,e^{-st}\,dt = \sum_{n=-\infty}^{\infty} f(nT)\,e^{-snT} = \sum_{n=-\infty}^{\infty} f(nT)\,z^{n}$$

So a second way to write $F^*(s)$ is

$$\boxed{F^*(s) = \sum_{n=-\infty}^{\infty} f(nT)\,e^{-snT}}$$ impulse train sampling

<u>Nyquist criterion</u>:

If $F(\omega)$ is bandlimited so that $F(\omega) = 0$, $|\omega| \geq \frac{\Omega}{2}$, then $f(t)$ can be recovered from $f^*(t)$ if samples are taken at least every $T = \frac{2\pi}{\Omega}$ seconds. In this case,



This criterion insures that the various components of the spectrum do not overlap. Thus, an ideal filter centered about $\omega = 0$ and cutting off at $\pm \Omega/2$ would restore our original spectrum, $F(\omega)$, exactly.

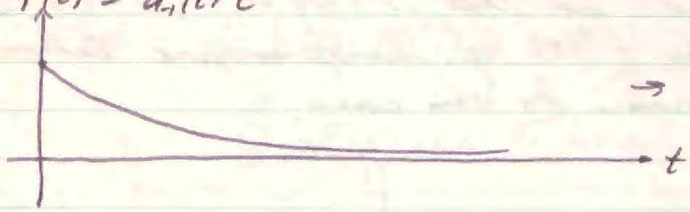
Note that we do not really have to use an ideal impulse train to get this result. The spectrum $\Pi(s)$ for any narrow pulse will be quite flat over the first several harmonics & we can still use our ideal low pass filter on $F(\omega)$ to pick off the "DC" component which will be virtually the same as for ideal impulse sampling.

Note also that we can add to $f(t)$ any waveform which is zero at the sample points, such as $\sin \Omega t$, etc., and we will get the same samples $f^*(t)$. In such a case, however, we are violating the band limiting criterion set above and we cannot expect to <u>exactly</u> recover $f(t)$.

In particular, since $F^*(s) = \sum_{-\infty}^{\infty} F(s - jk\Omega)$, any function having poles at the same abscissa, as $F(\omega)$, will give the same $F^*(\omega)$ unless the residues just happen to cancel. E.g., we couldn't differentiate:

Example:

$f(t) = u_1(t)\, e^{-at}$



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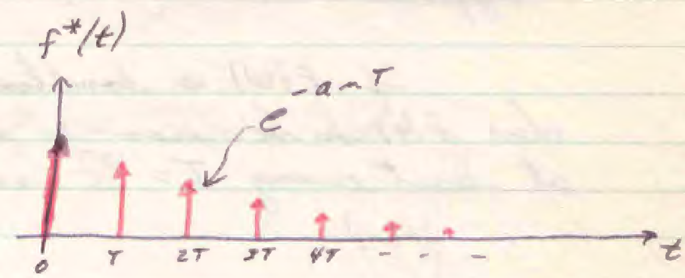
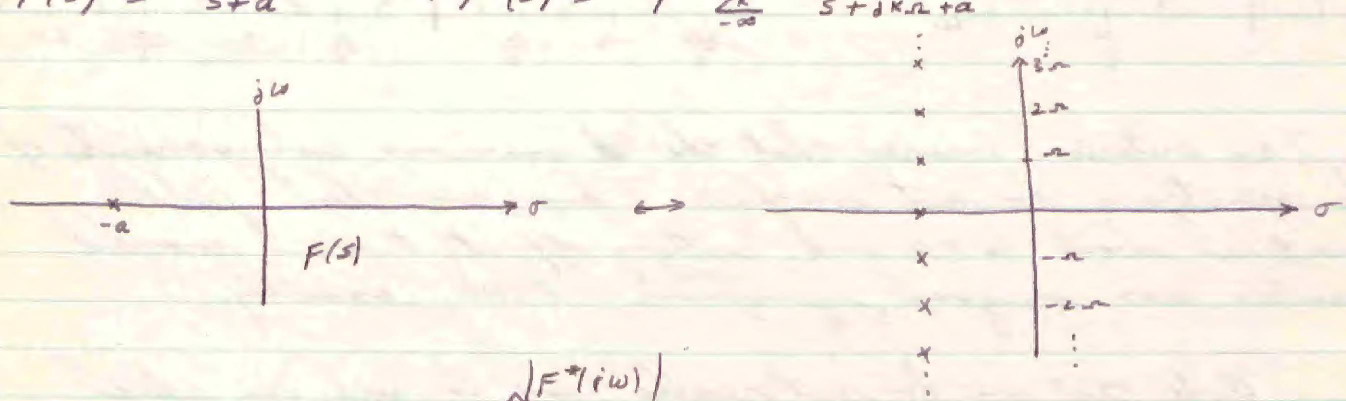
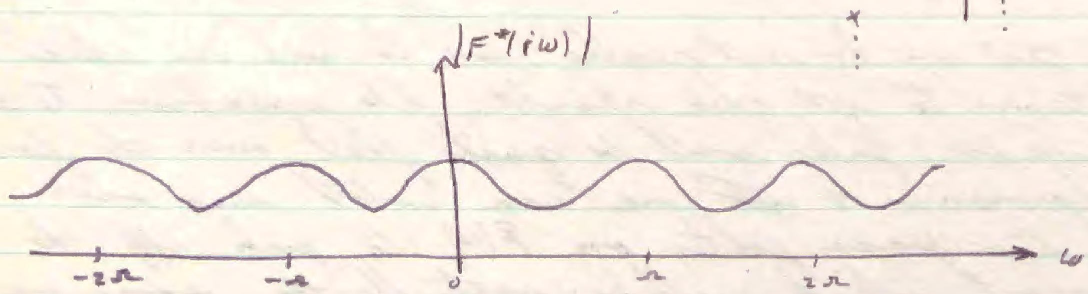


$$F(s) = \frac{1}{s+a} \quad \leftrightarrow \quad F^*(s) = \frac{1}{T}\sum_{-\infty}^{\infty} \frac{1}{s+jk\Omega+a}$$





Here, we have sampled a function that is not band limited. Thus $F^*(j\omega)$ is complicated & we cannot recover $F(t)$.

Another way to get this is

$$F^*(s) = \sum_0^\infty f(nT)e^{-snT} = -\frac{1}{2} + \sum_0^\infty e^{-anT}e^{-snT} = -\frac{1}{2} + \frac{1}{1-e^{-(s+a)T}}$$

$$= \frac{1}{1-e^{-aT}e^{-sT}}$$

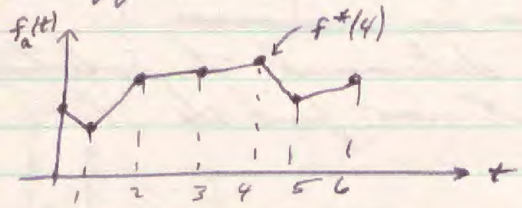
Near $s=-a$, it blows up; is this a pole? $\frac{1}{1-e^{-(s+a)T}} \to \frac{1}{1-[1-(s+a)T]} = \frac{1}{(s+a)T}$

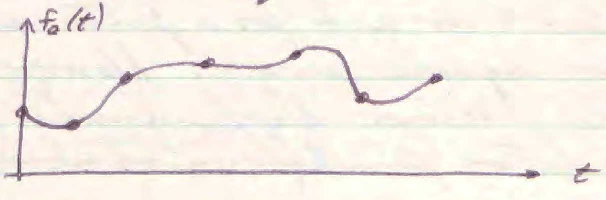
So it is a pole

Note $F^*(s) = \frac{1}{1-e^{-aT}e^{-sT}} \leftrightarrow F^*(z) = \frac{1}{1-e^{-aT}z^{-1}}$, $z = e^{-sT}$

Physical recovery of $f(t)$ from $f^*(t)$:

We can fit polynomial curves to points representing $f^*(t)$ as an approximation to $f(t)$. For example, straight lines or parabolic arcs:



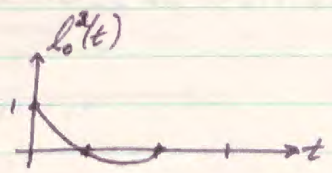


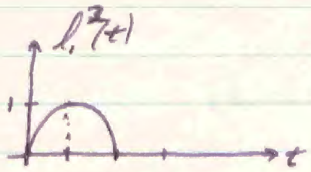
In general, we can use the arcs:

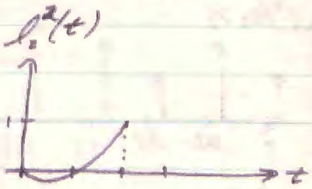
$$l_n^N(t) = \frac{(t-t_0)\cdots(t-t_N)}{(t_n-t_0)\cdots(t_n-t_N)}$$

omitting the n^{th} term.

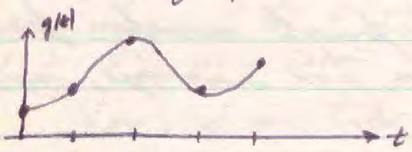
For example:







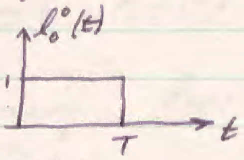
The set $\{l_n^N(t)\}$ can be used to fit an N^{th} order arc through $N+1$ points. For example, the curve fitting the five points below is



$$g(t) = g(0)\,l_0^4(t) + g(1)\,l_1^4(t) + g(2)\,l_2^4(t) + g(3)\,l_3^4(t) + g(4)\,l_4^4(t).$$

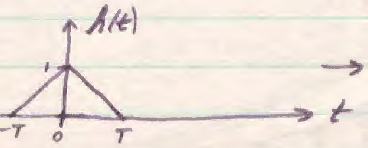
This equation is linear in $g(n)$. Thus we should be able to build a filter that would fit a polynomial to the points represented by the area of the impulses of $f^*(t)$:

$\underline{N=0} \Rightarrow$



$$\longleftrightarrow \quad \frac{1-e^{-sT}}{s}$$

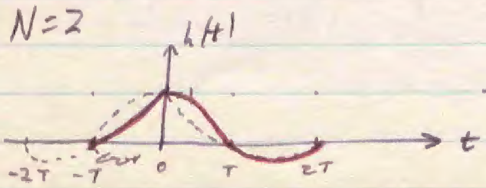
$\underline{N=1}$



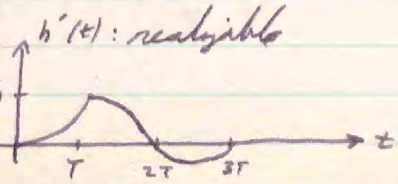
$\rightarrow$ $h'(t)$: realizable: had to accept delay to get fit.

$$H(s) = \frac{1}{Ts^2}\left[1-e^{-sT}\right]^2$$

$N=2$



$\rightarrow$ $h'(t)$: realizable

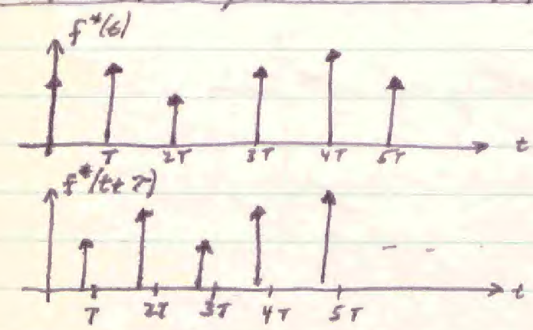


Besides being hard to synthesize, these higher order filters tend to be oscillatory. Generally we would want to stick with $N=0$ or $N=1$.

Using Wiener filters to get $f(t)$ from $f^*(t)$:

Ideally, we would like to build a filter which will give us $f(t)$ out when we put $f^*(t)$ in. Of course we cannot do this & so ask for a filter which gives us minimum mean square error. Having already determined the solution to the Wiener-Hopf equation, we must now find the correlation functions involved:

Given $\varphi_{ff}(\tau)$, what is $\varphi_{f^*f^*}(\tau)$?



$$\varphi_{f^*f^*}(\tau) = \frac{1}{2T}\int_{-T}^{T} f^*(t)\, f^*(t+\tau)\, dt$$

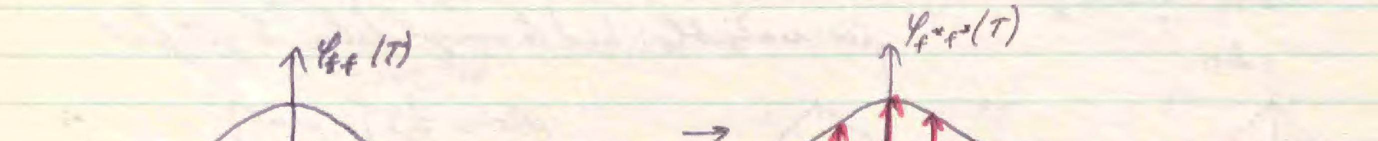
When $\tau = KT$, clearly we will get zero.
When $\tau = KT$, we will get an impulse with area $\lim_{N\to\infty} \frac{1}{2N+1}\sum_{-N}^{N} f(nT)\,f(nT+kT)$. And of course, an infinite number of discrete points averaged is as good as a continuous time average, so this area is just $\varphi_{ff}(KT)$

$\cancel{\varphi_{f^*f^*}(\tau)}$

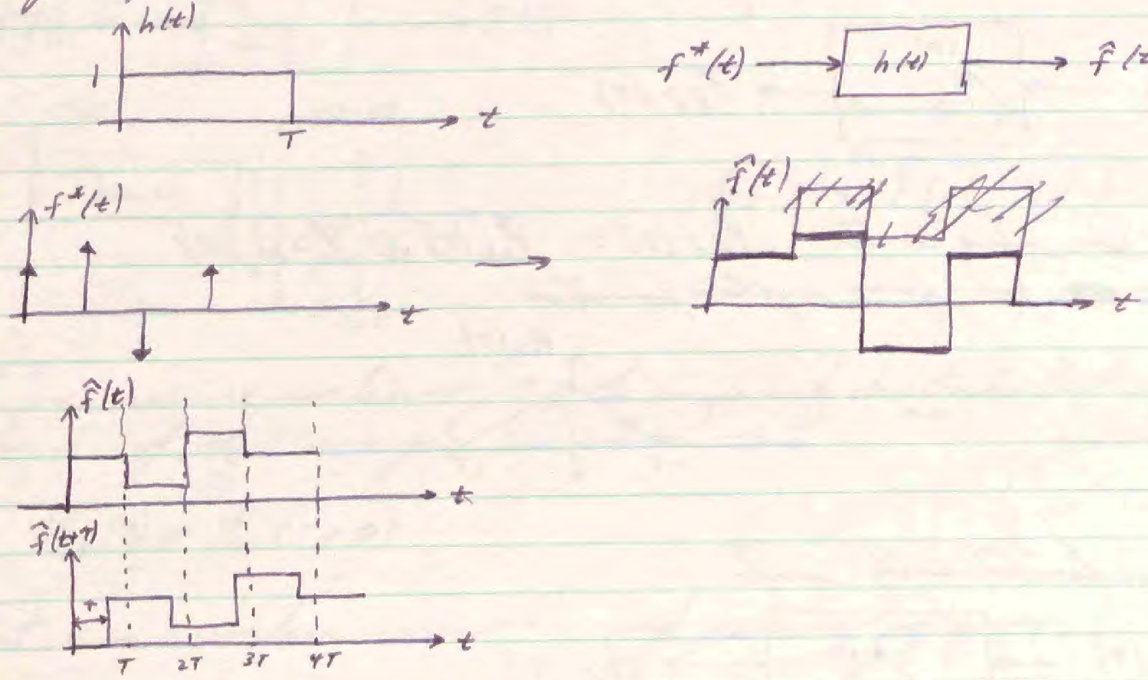
Thus,

$$\varphi_{f^*f^*}(\tau) = \varphi_{ff}(\tau)\sum_{k=-\infty}^{\infty} \delta(\tau - KT)$$

or $$\boxed{\varphi_{f^*f^*}(\tau) = \varphi_{ff}^*(\tau)}$$



Another way of getting at this is to consider putting $f^*(t)$ through a filter with impulse response $h(t)$; giving output $\hat{f}(t)$.

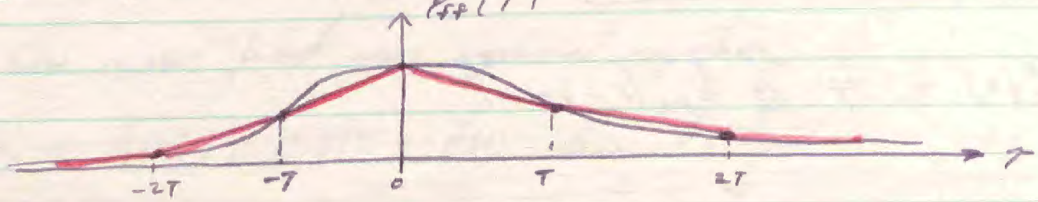


Consider that part of time $T<t<2T$, $0<\tau<T$.
Now, for ~~[illegible]~~ $\tau+T<t<2T$, both $\hat{f}(t)$ and $\hat{f}(t+\tau)$ are equal to $f(T)$. In the interval $T<t<T+\tau$, $\hat{f}(t)=f(T)$, but $\hat{f}(t+\tau)=f(0)$. Hence, in this second interval $\overline{\hat{f}(t)\,\hat{f}(t+\tau)}=\varphi_{ff}(T)$

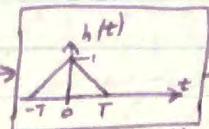
Now, if we take a time average $\overline{\hat{f}(t)\,\hat{f}(t+\tau)}$, ~~[illegible]~~ and the source is ergodic, we will clearly get $\varphi_{ff}(0)$ part of the time & $\varphi_{ff}(T)$ part of the time. In particular,

$$\varphi_{\hat{f}\hat{f}}(\tau) = \varphi_{ff}(0)\left[\frac{T-\tau}{T}\right] + \varphi_{ff}(T)\left[\frac{\tau}{T}\right], \quad 0<\tau<T$$

or, more generally, $\varphi_{\hat{f}\hat{f}}(\tau)$ is a piecewise continuous straight line fit to $\varphi_{ff}(\tau)$

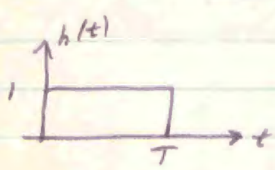


But this is the result we would get by putting ~~[illegible]~~ $\varphi_{ff}^*(\tau)$ through the filter ~~with~~ $N=1$ on page 57:

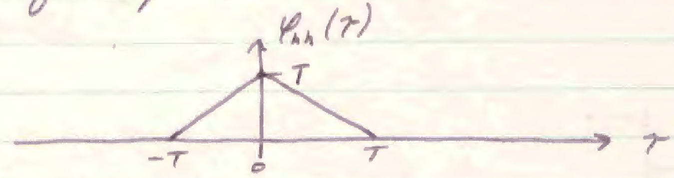
$\varphi_{ff}^*(\tau) \longrightarrow$  $\longrightarrow \varphi_{\hat{f}\hat{f}}(\tau)$

(1)

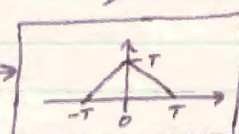
But now we can also write $\varphi_{\hat{f}\hat{f}}(\tau) = \varphi_{hh}(\tau) \circledast \varphi_{f^*f^*}(\tau)$
where $h(t)$ is ~~a~~ our given system function



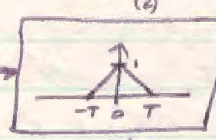
$\longrightarrow$



So, we can schematically write

$\varphi_{f^*f^*}(\tau) \longrightarrow$  $\longrightarrow \varphi_{\hat{f}\hat{f}}(\tau)$

(2)

or $\quad T\varphi_{f^*f^*}(\tau) \longrightarrow$  $\longrightarrow \varphi_{\hat{f}\hat{f}}(\tau)$

(3)

The system functions (3) and (1) are identical, as are their outputs. Thus we conclude that they must have identical inputs:

$$\varphi_{ff}^*(\tau) = T\,\varphi_{f^*f^*}(\tau)$$

or

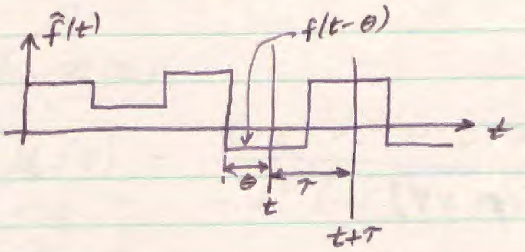
$$\boxed{\varphi_{f^*f^*}(\tau) = \frac{1}{T}\,\varphi_{ff}^*(\tau)}$$

and since $\quad \Phi_{ff}^*(s) = \frac{1}{T}\sum_{-\infty}^{\infty}{}_{k}\, \Phi_{ff}(s+jk\Omega)$,

then

$$\boxed{\Phi_{f^*f^*}(s) = \frac{1}{T^2}\sum_{k=-\infty}^{\infty} \Phi_{ff}(s+jk\Omega)}$$

<u>Now then, what is $\varphi_{f^*f}(\tau)$?</u>



$$\varphi_{\hat{f}f}(\tau) = \overline{\hat{f}(t)\, f(t+\tau)}$$

If we assume we know θ, the uniformly distributed random "phase" of this sample function, then

$$\varphi_{\hat{f}f}(\tau) = \varphi_{ff}(\tau+\theta)$$

But θ is a random variable, and averaging gives

$$\varphi_{\hat{f}f}(\tau) = \int_0^T \varphi_{ff}(\tau+\theta)\, P(\theta)\, d\theta = \frac{1}{T}\int_0^T \varphi_{ff}(\tau+\theta)\, d\theta \longleftarrow$$

But $\varphi_{\hat{f}f}(\tau) = \overline{\hat{f}(t)\, f(t+\tau)} = \overline{\left[\int_0^T h(x)\, f^*(t-x)\, dx\right] f(t+\tau)}$

$$= \int_0^T \overline{f^*(t-x)\, f(t+\tau)}\, dx = \int_0^T \varphi_{f^*f}(\tau+x)\, dx \longleftarrow$$

Equating the two checked results gives

$$\frac{1}{T}\int_0^T \varphi_{ff}(\tau+\theta)\, d\theta = \int_0^T \varphi_{f^*f}(\tau+\theta)\, d\theta \quad \text{for all } \tau$$

Thus $\boxed{\varphi_{f^*f}(\tau) = \frac{1}{T}\varphi_{ff}(\tau)}$

We can now find our optimal system:

$f^*(t) \longrightarrow \boxed{H_0(s)} \longrightarrow g(t) \leftrightarrow d(t) =$ desired output

$$\boxed{H_0(s) = \frac{1}{\Phi^+_{f^*f^*}(s)}\left[\frac{\Phi_{f^*d}(s)}{\Phi^-_{f^*f^*}(s)}\right]_+ = \frac{1}{\Phi_{f^*f^*}(s)}\left[\frac{\frac{1}{T}\Phi_{ff}(s)}{\Phi^-_{f^*f^*}(s)}\right] \text{ if } d(t) = f(t)}$$

Example:

Suppose we know $\Phi_{ff}(s) = \frac{1}{1-s^2} \leftrightarrow \varphi_{ff}(\tau) = \frac{1}{2}e^{-|\tau|}$

and we want $d(t) = f(t-nT)$:

First we find $\Phi_{f^*f^*}(s) = \frac{1}{T}\Phi_{ff}^*(s)$

But $\Phi_{ff}^*(s) \leftrightarrow \varphi_{ff}^*(\tau) = \frac{1}{2}\sum_{-\infty}^{\infty} e^{-|k|T}\,\delta(\tau - kT)$

So $\Phi_{ff}^*(s) = \frac{1}{2}\sum_{0}^{\infty} e^{-kT}z^k + \frac{1}{2}\sum_{-1}^{-\infty} e^{kT}z^k$, $z \equiv e^{-sT}$

$$= \frac{1}{2}\,\frac{1}{1-e^{-T}z} + \frac{1}{2}\,\frac{1}{1-e^{-T}\frac{1}{z}} = \frac{\frac{1}{2}(1-e^{-2T})}{(1-e^{-T}z)(1-e^{-T}\frac{1}{z})}$$

or $\Phi_{f^*f^*}(s) = \frac{a^2/T}{(1-e^{-T}z)(1-e^{-T}z^{-1})}$

Now, $d(t) = f(t-nT)$ so $\varphi_{f^*d}(\tau) = \varphi_{f^*f}(\tau - nT)$

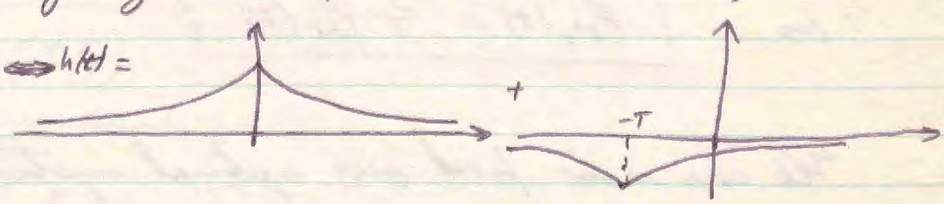
or $\Phi_{f^*d}(s) = \frac{1}{T}\Phi_{ff}(s)\,z^n = \frac{z^n}{T(1-s^2)}$

Thus

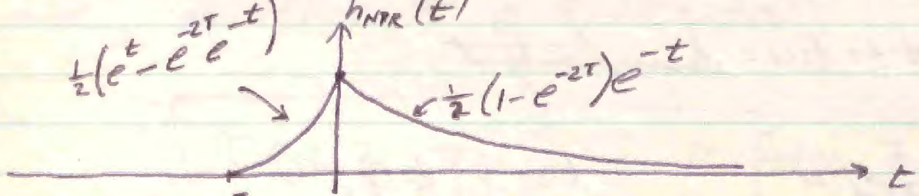
$$H_0(s) = \frac{(1-e^{-T}z)}{a^2}\left[\frac{z^n(1-e^{-T}\frac{1}{z})}{1-s^2}\right]_+$$

If we ignore the realizability right now, and let $n=0$,

$H_0(s) \to \frac{1}{1-s^2} - \frac{e^{-T}z^{-1}}{1-s^2} \Leftrightarrow h(t) =$

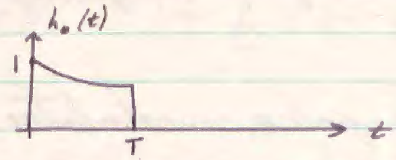


or



Considering realizability, first let $n = 0$:

$$H_0(s) \rightarrow \frac{1-e^{-T}z}{1+s} \quad \leftrightarrow$$



If $n \geq 1$:

$$H_0(s) = \frac{(1-e^{-T}z)(1-e^{-T}z^{-1})}{a^2(1-s^2)} = \left[\frac{1-e^{-T(s+1)}}{a(1+s)}\right]\left[\frac{1-e^{-T(-s+1)}}{a(1-s)}\right]$$

We can find $h_0(t)$ by convolution:

<u>Flow graph techniques for sampling:</u>

$\circ \xrightarrow{M} \circ \equiv$ impulse modulator, sampling rate $= \frac{1}{T}$

$\circ \xrightarrow{2M} \circ \equiv$ " " , " " $= \frac{2}{T}$

$\circ \xrightarrow{3M} \circ \equiv$ " " , " " $= 3/T$

$\circ \xrightarrow{K(s)} \circ \equiv$ a linear system with transfer function rational in s.

$\circ \xrightarrow{D(z)} \circ \equiv$ a linear system with transfer function rational in $z = e^{-sT}$.

$\circ \xrightarrow{H(s)} \circ \equiv$ a linear system that cannot be broken into K's and D's; e.g., $\frac{1}{s-z}$.

The element M corresponds to an operation we know, but there is no transfer function that will represent this element. M is not commutative, but it is linear.

<u>Simple flow graphs</u>:

Consider the combination of elements:

$$F(s) \circ \xrightarrow{M} \circ \xrightarrow{D(z)} \circ \xrightarrow{K(s)} \circ \quad G(s) = F^*(s)\, D(z)\, K(s)$$

We cannot find an expression for finding $G(s)$ exactly, so we ask what is $g(t)$ at the sample times?

$$F(s) \circ \xrightarrow{M} \circ \xrightarrow{D(z)} \circ \xrightarrow{K(s)} \circ \xrightarrow{M} \circ \quad G^*(s) = [F^*(s)\, K(s)\, D(z)]^*$$

$$\left[F^*(s)\,D(z)\,K(s)\right]^* = \frac{1}{T}\sum_{k=-\infty}^{\infty} K(s+jk\Omega)\,D\left(e^{-(s+jk\Omega)T}\right)\frac{1}{T}\sum_{m=-\infty}^{\infty} F(s+jk\Omega+jm\Omega)$$

Now

$$D\left[e^{-(s+jk\Omega)T}\right] = D\left[e^{-sT}e^{-jk\Omega T}\right] = D\left(z e^{-jk2\pi}\right) = D(z)$$

$$\frac{1}{T}\sum_{m=-\infty}^{\infty} F(s+jk\Omega+jm\Omega) = \frac{1}{T}\sum_{m=-\infty}^{\infty} F(s+jm\Omega) = F^*(s)$$

So $G^*(s) = \frac{1}{T}\sum_{k=-\infty}^{\infty} K(s+jk\Omega)\,D(z)\,F^*(s) = F^*(s)\,D(z)\,\frac{1}{T}\sum_{k=-\infty}^{\infty} K(s+jk\Omega)$

$$\boxed{\left[F^*(s)\,D(z)\,K(s)\right]^* = F^*(s)\,D(z)\,K^*(s)}$$

If we now double the sample rate of the second sampler, we get

$$F(s) \xrightarrow{M} \circ \xrightarrow{D(z)} \circ \xrightarrow{K(s)} \circ \xrightarrow{2M} \left[F^*(s)\,D(z)\,K(s)\right]^{**}$$

$$\left[F^*(s)\,D(z)\,K(s)\right]^{**} = \frac{2}{T}\sum_k K(s+jk2\Omega)\,D\left(z e^{-jk4\pi}\right)\frac{1}{T}\sum_m F(s+j2k\Omega+jm\Omega)$$

$$\boxed{= F^*(s)\,D(z)\,K^{**}(s) = \left[F^*(s)\,D(z)\,K(s)\right]^{**}}$$

Obviously,

$$F(s) \xrightarrow{M} \circ \xrightarrow{K} \circ \xrightarrow{nM} G(s)$$

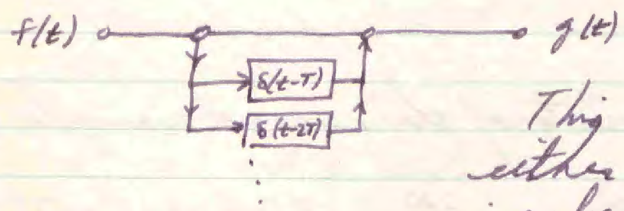
$$\boxed{\left[F^*(s)\,K(s)\right]^{(n*)} = F^*(s)\,K^{(n*)}(s)}$$

Physical interpretation of discrete systems:

We have defined a discrete system mathematically as a system whose transfer function is rational in $Z = e^{-sT}$. If we use this transfer function as a function of s, we find an exact correspondence with a system whose impulse response is an impulse train:

$$F(s) = \int f(t) e^{-st} dt = \int \sum u(t-nT) f(n) e^{-st} dt = \sum f(n) e^{-snT} = \sum f(n) Z^n .$$

Thus we can interpret a discrete system in the continuous time domain as one made up of delay lines.



This type of discrete system can have either continuous inputs or impulse train inputs.

Another way of building a discrete system would be to use a digital computer. Here the discrete signals would be "numbers" rather than impulsive areas as in the previous model. It might be easier to build a discrete system in this way, but it would be more difficult to tie into continuous systems.

Example: $F(s) \xrightarrow{M} f^*(t) \xrightarrow{K(s)} g(t) \xrightarrow{M} g^*(t)$ $\quad G^*(s) = F^*(s) K^*(s)$

$$F^*(s) = \frac{TZ}{1-Z} \quad ; \quad K(s) = \frac{a}{s(s+a)}$$

$$K^*(s) = \frac{Z(1-e^{-aT})}{(1-Z)(1-e^{-aT}Z)}$$

So

$$G^*(s) = \frac{z^2(1-e^{-aT})}{(1-z)^2(1-e^{-aT}z)} = \frac{Tz}{(1-z)^2} - \frac{T}{1-e^{-aT}}\left[\frac{1}{1-z} - \frac{1}{1-e^{-aT}z}\right]$$

$$g^*(t) = nT - \left(\frac{T}{1-e^{-aT}}\right)\left(1-e^{-anT}\right)$$

Now consider the same system, but with the last sampler as a double rate sampler:

> We can get $K^{**}(s)$ from $K^*(s)$ by replacing T with $T/2$. We can do this when no cancellations occur in the summation for $K^*(s)$, and this will be so if the poles of $K(s)$ lie between $-j\frac{\Omega}{2}$ and $+j\frac{\Omega}{2}$.

$$K^*(s) = \frac{e^{-sT}(1-e^{-aT})}{(1-e^{-sT})(1-e^{-aT}e^{-sT})}$$

$$K^{**}(s) = \frac{e^{-\frac{sT}{2}}(1-e^{-aT/2})}{(1-e^{-sT/2})(1-e^{-aT/2}e^{-sT/2})}$$

$$G^{**}(s) = \frac{T(e^{-\frac{sT}{2}})^3(1-e^{-aT/2})}{(1-e^{-sT/2})(1-e^{-aT/2}e^{-sT/2})(1+e^{-sT/2})} = F^*(s)K^{**}(s)$$

The pole $\frac{1}{1+e^{-sT/2}}$ corresponds to a "ripple term."

We can partial fraction expand $G^{**}(s)$ to get the residue of this pole:

$$G^{**}(s) = \frac{-\frac{T(1-e^{-\frac{aT}{2}})}{4(1+e^{-aT/2})}}{1+y} + \frac{R}{(1-y)^2(1-e^{-\frac{aT}{2}}y)}$$

We know exactly the time function corresponding to this last term. The first term gives us the frequency of the "ripple" that is added to our 1x sampler solution:

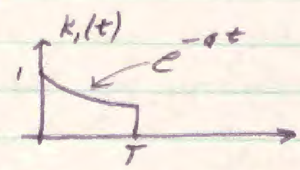
$$g(t) = t - \frac{T}{1-e^{-aT}}\left[1-e^{-at}\right] + \text{"ripple term"}$$

We want the ripple term to vanish at $t = kT$. We do not know the exact nature of $g(t)$ at all points, but we can make an approximation that will yield exact results at sample instants

$$g(t) \approx t - \frac{T}{1-e^{-aT}}\left[1-e^{-at}\right] + \frac{T(1+e^{-aT/2})}{8(1-e^{-aT/2})}\left(1-\cos\Omega t\right)$$

Trembath's procedure:

Suppose we have $D(z)K(s)$ and want to evaluate the impulse response $\leftrightarrow d(t) * k(t)$:

Let $K_1(s) \leftrightarrow$ 

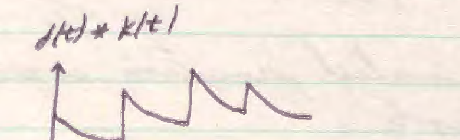
if $K(s) = \frac{1}{s+a}$

Then $K_1(s) = \dfrac{1-e^{-aT}e^{-sT}}{s+a}$

$$D_1(z)K_1(s) = D(z)K(s) \Rightarrow D_1(z) = \frac{D(z)}{1-e^{-aT}e^{-sT}} = \frac{D(z)}{(1-e^{-aT}z)}$$

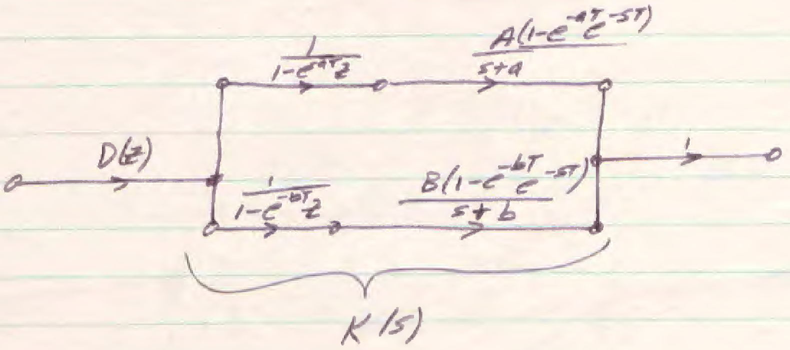
Then if

then

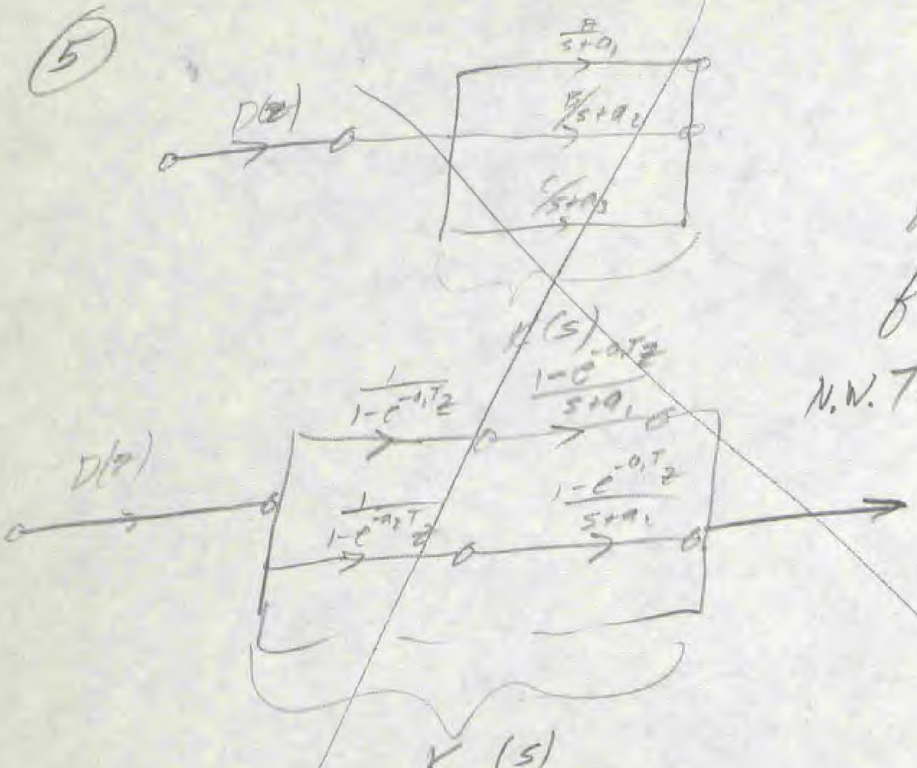


For more general $K(s)$, we can partial fraction $K(s)$

$$K(s) = \frac{A}{s+a} + \frac{B}{s+b} + \cdots$$



(5)

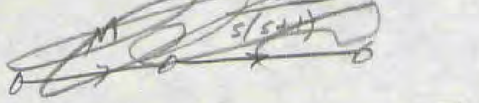


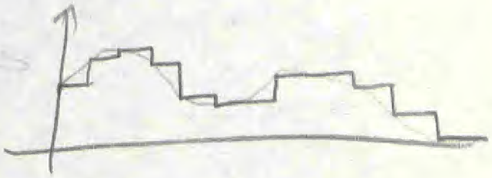
only well for finding exactly;

N.W. Trembath's procedure.

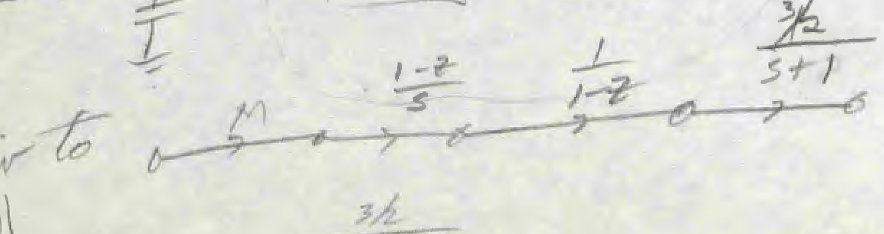
~~Multiple poles~~

E.g.:

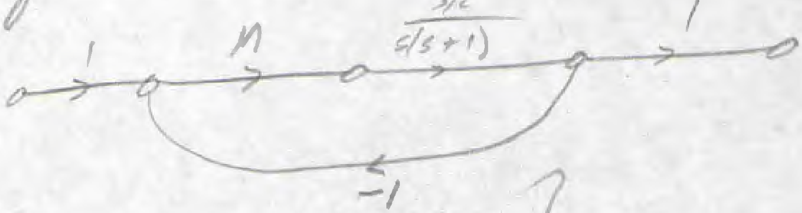




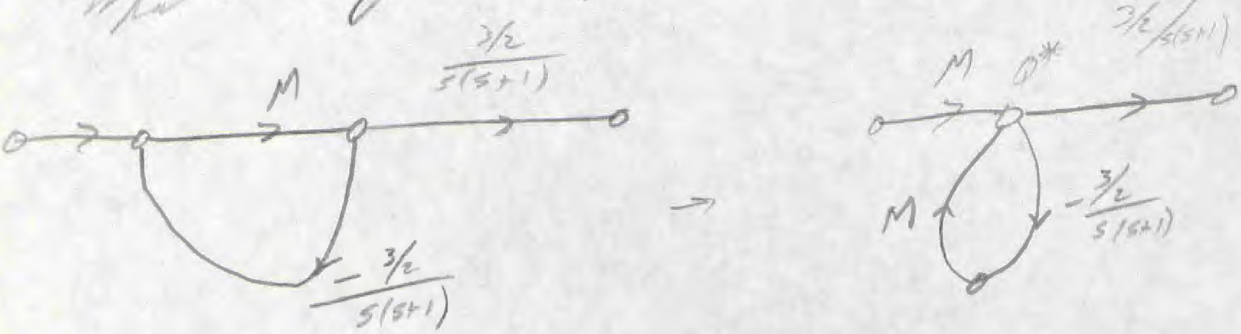
equiv to



(6) put feedback around.



What is resp to step?



$Q^* \;\circ \xrightarrow{-K} \circ \xrightarrow{M} \circ\; Q^*K^* = \cancel{0}$

$Q^* = F^* - Q^*K^*$

$$\frac{Q^*(s)}{F^*(s)} = \frac{1}{1+K^*}$$

$$G(s) = F^*(s)\,\frac{1}{1+\left(\frac{3/2}{s(s+1)}\right)^*}\,\frac{3/2}{s(s+1)}$$

See if can find $G^*(s)$ & $G^{**}(s)$

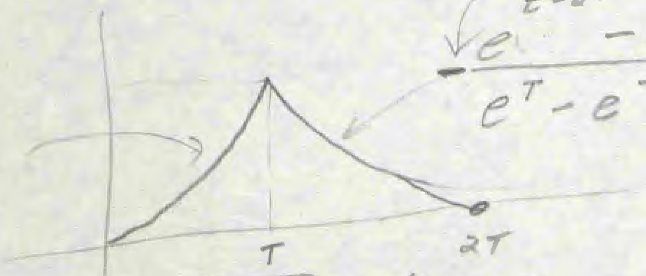
$g^*(t)$ & $g^{**}(t)$

Will do in class next time.

6.54

Notes correction:

$$\frac{e^{t}-e^{-t}}{e^{T}-e^{-T}} \qquad -\frac{e^{t-2T}-e^{2T-t}}{e^{T}-e^{-T}}$$

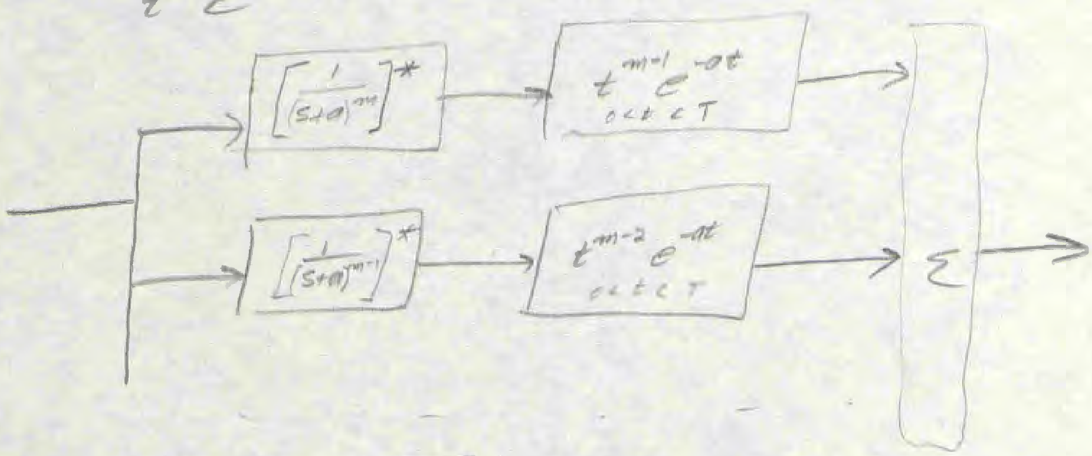


for prob #3

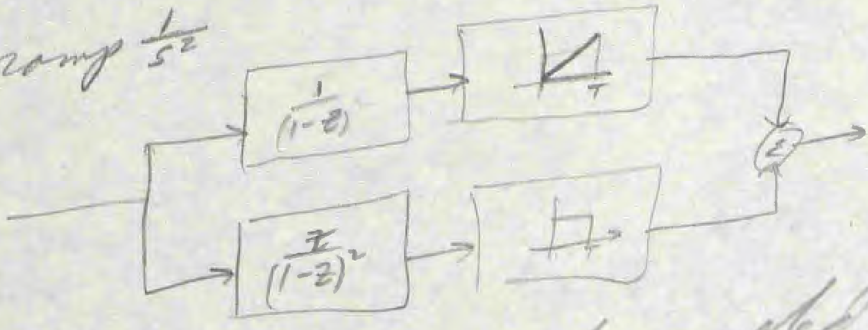
$\boxed{n=1}$ not $n=2$

$$\frac{1}{(s+a)^m} \longleftrightarrow t^m e^{-at}$$

$$\frac{1}{(s+a)^m} \equiv$$



e.g., ramp $\frac{1}{s^2}$



Methods for finding resp of sampled systs

(1) Basic rate sampling

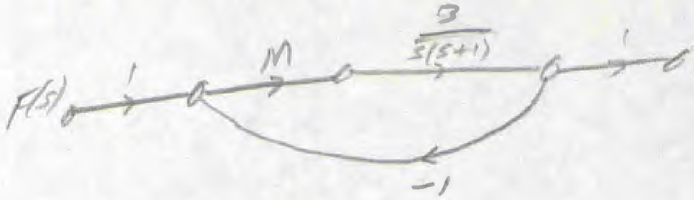
(2) Multiple rate "

(3) Convolution techniques (Trembath's method)

(4) Approximation techniques

(2)

<u>Eg_2 :</u>



$$G(s) = \left[\frac{1}{1+\left(\frac{3}{s(s+1)}\right)^*}\right]\frac{3F^*(s)}{s(s+1)}$$

$$\cancel{G(s) = \frac{(1-z)(1-\frac{1}{2}z)}{1+\frac{1}{2}z^2}}$$

$$\left[\frac{3}{s(s+1)}\right]^* = \left[\frac{3}{s} - \frac{3}{s+1}\right]^* = \frac{3}{1-z} - \frac{3}{1-e^{-T}z}$$

$\boxed{T \text{ chosen so that } e^T = 2}$

$$= \frac{3z}{(1-z)(1-e^{-T}z)}$$

$$= \frac{\frac{3}{2}z}{(1-z)(1-\frac{1}{2}z)}$$

So $\quad G(s) = F^*(s)\,\dfrac{(1-z)(1-\frac{1}{2}z)}{(1+\frac{1}{2}z^2)} \cdot \dfrac{3}{s(s+1)}$

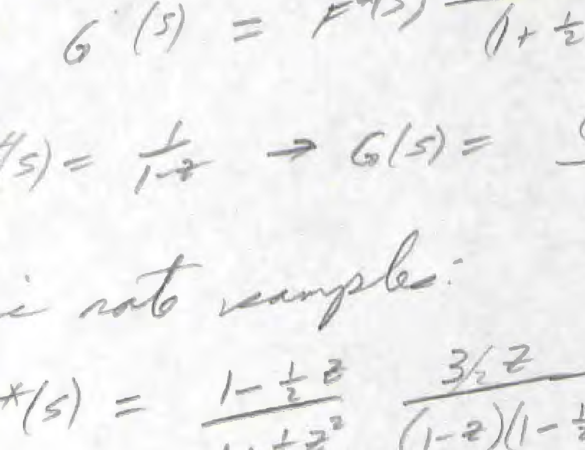
$F^*(s) = \dfrac{1}{1-z} \rightarrow G(s) = \dfrac{(1-\frac{1}{2}z)}{1+\frac{1}{2}z^2}\cdot\dfrac{3}{s(s+1)}$

Basic rate samples:

$$G^*(s) = \frac{1-\frac{1}{2}z}{1+\frac{1}{2}z^2}\;\frac{3/2\,z}{(1-z)(1-\frac{1}{2}z)} = \frac{3/2\,z}{(1-z)(1+\frac{1}{2}z^2)}$$

$$= \frac{2}{1-z} + \frac{\frac{1}{2}z-1}{1+\frac{1}{2}z^2} = \frac{2}{1-z} - 1 + \tfrac{1}{2}z + \tfrac{1}{2}z^2 - \tfrac{1}{4}z^3 - \tfrac{1}{4}z^4 + \tfrac{1}{8}z^5 \ldots$$

$g^*(k)$



(3)

Double rate sampler :

$$\left[\frac{3}{s(s+1)}\right]^{**} = \frac{3}{1-y} - \frac{3}{1-\frac{1}{\sqrt{2}}y} = \frac{3y(1-\frac{1}{\sqrt{2}})}{(1-y)(1-\frac{1}{\sqrt{2}}y)}$$

$(1-\frac{1}{\sqrt{2}}y)(1+\frac{1}{\sqrt{2}}y)$

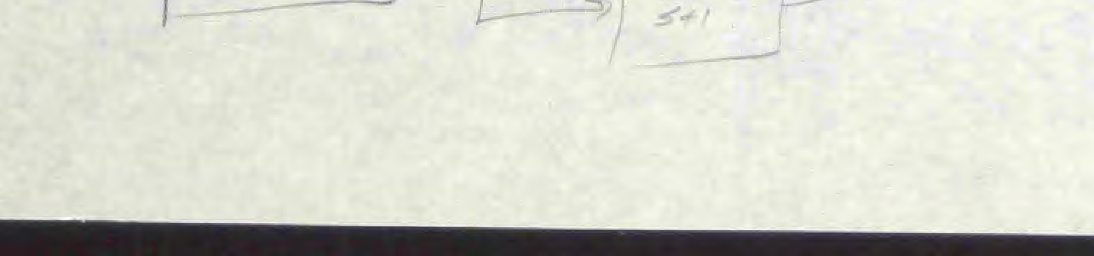
$$G^{**}(s) = \frac{1-\frac{1}{2}y^2}{1+\frac{1}{2}y^4} \quad \frac{3(1-\frac{1}{\sqrt{2}})y}{(1-y)(1-\frac{1}{\sqrt{2}}y)}$$

$$= \frac{3(1-\frac{1}{\sqrt{2}})y(1+\frac{1}{\sqrt{2}}y)}{(1-y)(1+\frac{1}{2}y^4)}$$

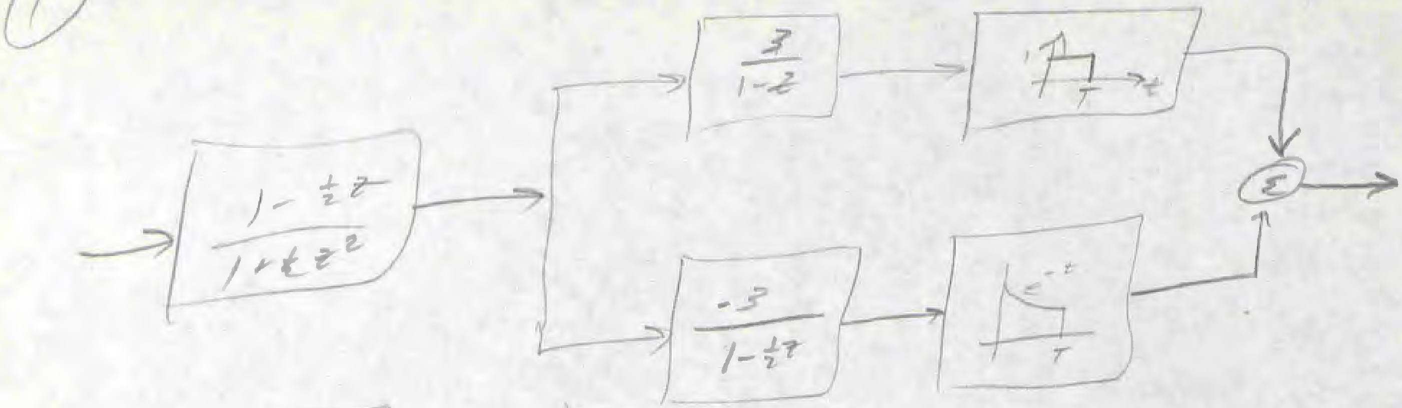
$$= \frac{1}{1-y} + \frac{-1+0.12y+\frac{1}{2}y^2+\frac{1}{2}y^3}{1+\frac{1}{2}y^4}$$

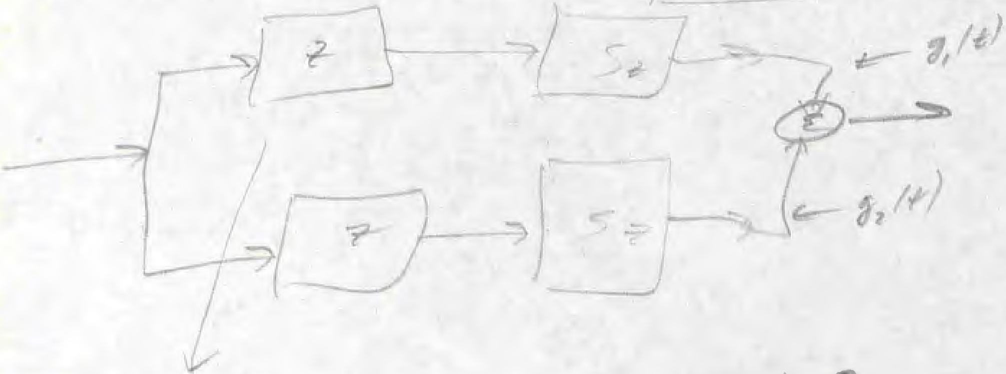
$$= \frac{1}{1-y} + \left(-1+0.12y+\frac{1}{2}y^2+\frac{1}{2}y^3\right)\left(1-\frac{1}{2}y^4+ - - -\right)$$

Trambath meth :



④

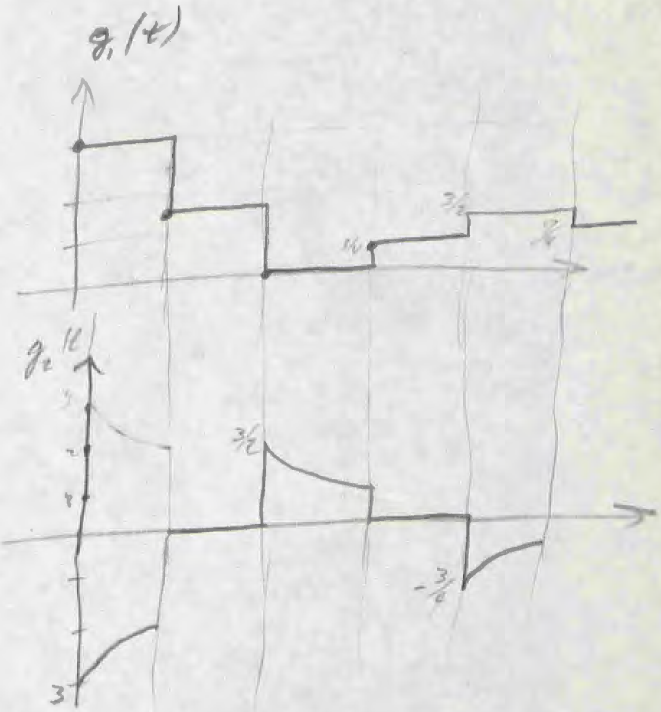




1162 =
111212112

$$\frac{3(1-\frac{1}{2}z)}{(1+\frac{1}{2}z^2)(1-z)} = \frac{1}{1-z} + \frac{+\frac{1}{2}z+2}{1+\frac{1}{2}z^2}$$

$$= \frac{1}{1-z} + 2 + \frac{1}{2}z - z^2 - \frac{1}{2}z^3 \rightarrow$$

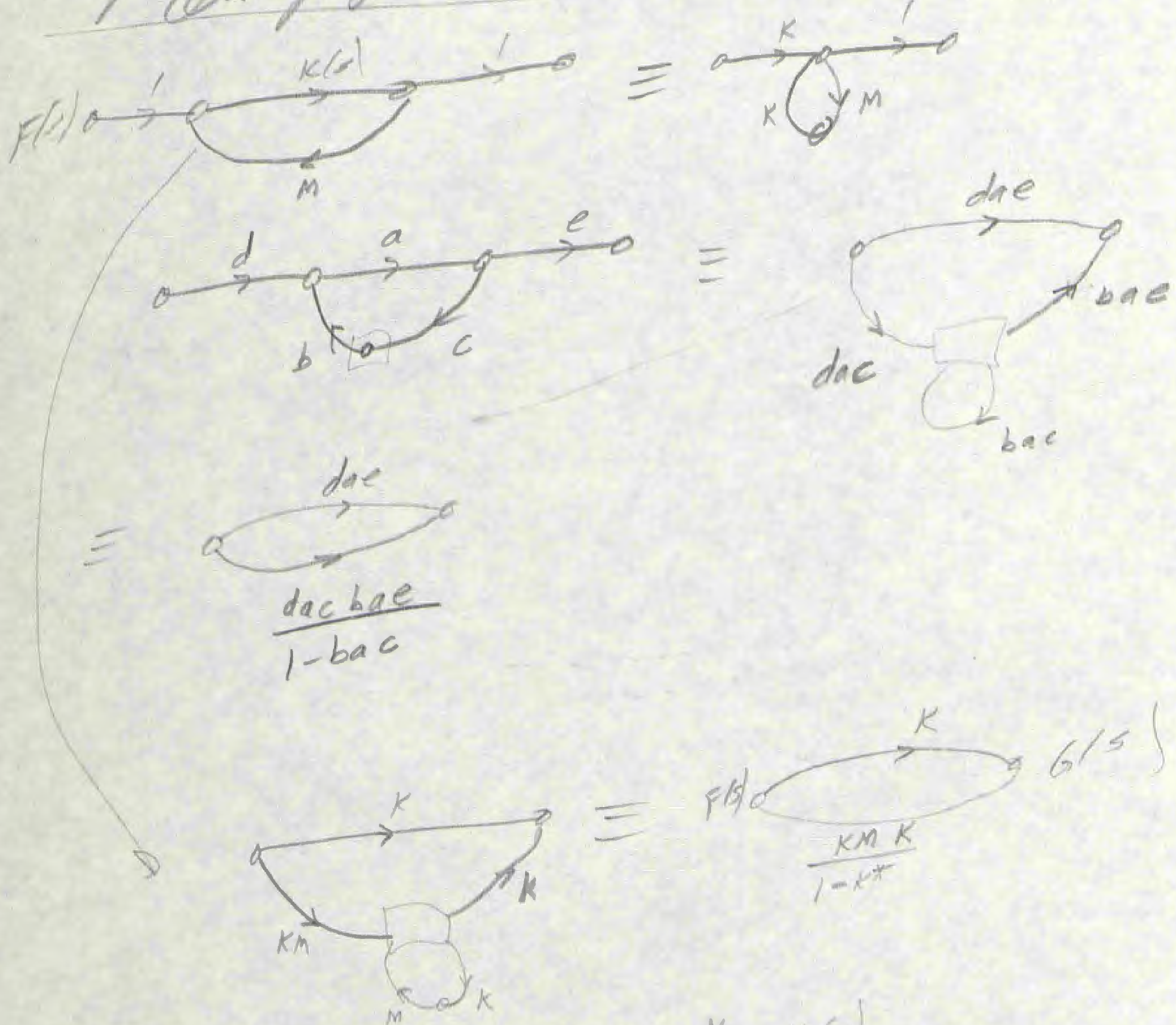


Now get $g_2(t)$:

$$\frac{-3(1-\frac{1}{2}z)}{(1+\frac{1}{2}z^2)(1-\frac{1}{2}z)} = -3\left[1-\frac{1}{2}z^2 + \cdots\right.$$

etc.

5 Flow graph red:

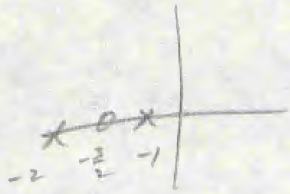


$$G(s) = F(s)K(s) + \left[F(s)K(s)\right]^{*} \frac{K(s)}{1-K^{*}(s)}$$

6.54 / Quiz 4 May

Prob sess Thurs 27 April (9-11)

$$F(s) = \frac{1}{s+1} + \frac{1}{s+2} = \frac{2s+2}{(s+1)(s+2)}$$



$\cancel{\frac{1}{s+2} = \frac{1/2}{1+s/2}}$

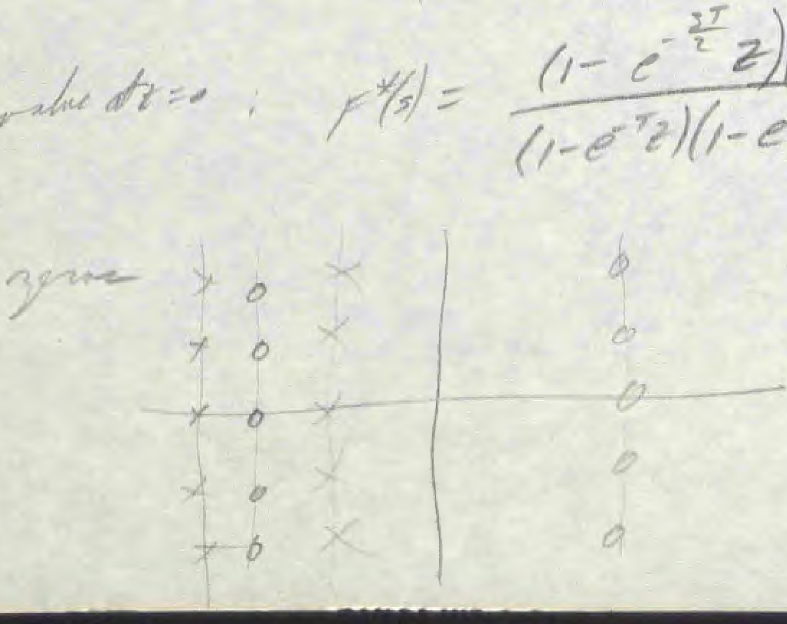
$$F^*(s) = \sum_{-\infty}^{\infty} F(s + jk\Omega)$$

only take ½ sample at ~~zero~~ $t = 0$

$$F^*(s) = \frac{1}{1-e^{-T}z} + \frac{1}{1-e^{-2T}z} - \frac{1}{2} - \frac{1}{2}$$

With whole sample at $t=0$: $$\frac{2-(e^{-T}+e^{-2T})z}{(1-e^{-T}z)(1-e^{-2T}z)}$$

w/ avg value at $t=0$: $$F^*(s) = \frac{(1-e^{-\frac{3T}{2}}z)(1+e^{-\frac{3T}{2}}z)}{(1-e^{-T}z)(1-e^{-2T}z)}$$



Can use either as long as always take value at discontinuity ~~at~~

(2)

Back to last class: flow graph reduction.

Choose resid nodes so all f. b. paths are broken.

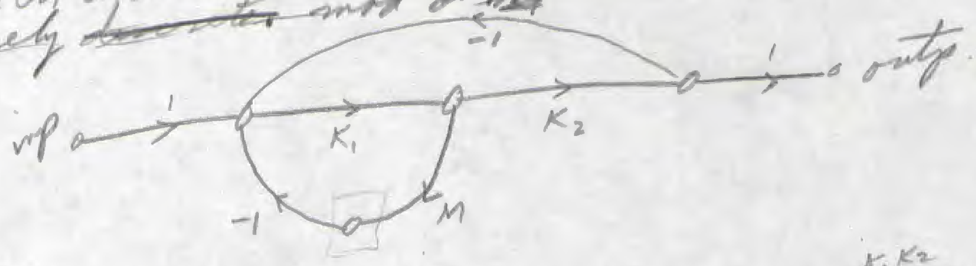
General Rule:

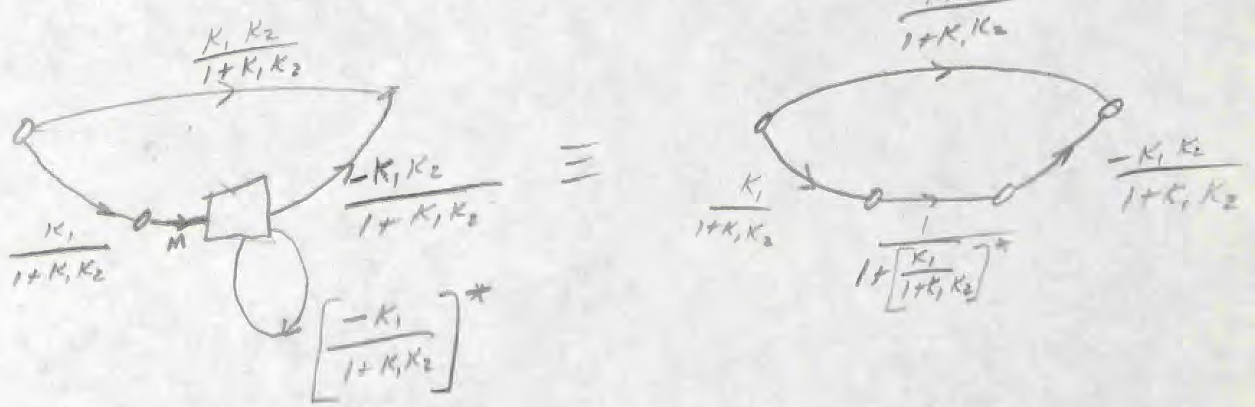
Pick the inp, outp, & a set of internal (residual) nodes which break all f.b. paths. If modulators are present in a loop, pick the ~~[illegible]~~ resid node as the outp of the mod. Not nec to break loops w/ only for fns.

May be nec to repeat procedure. Eventually get reduction to parallel paths.

Putting extra branches if nec so signal at resid nodes is purely ~~[illegible]~~ mod output.

Eg:

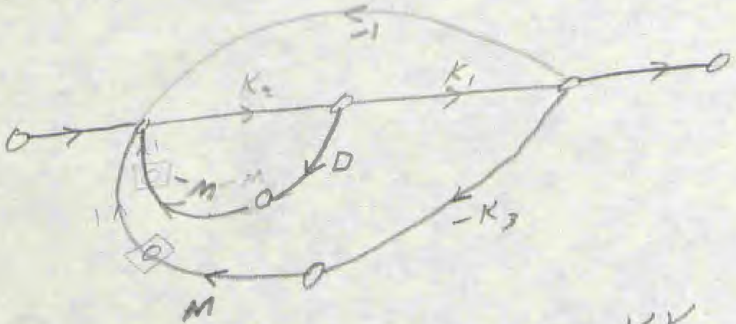




(3)

Tough Example:

R→



$$\left[(K_3K)^* D_b DKK_2 - KK_3K_2\right]$$

reduces to

$$K_1K_2 - KMDD_b\,KK_2 + \frac{\left[KK_2MD_a - KMDD_b(KK_2)^*D_a\right]\left[\ \right]}{1-(K_3K)^*DD_b(KK_2)^*D_a}$$

where

$$D_a = \frac{1}{1+(KK_2K_3)^*}$$

$$D_b = \frac{1}{1+K^*D}$$

$$K = \frac{K_1}{1+K_1K_2}$$

~~MKD~~

Irreducible [illegible]
(one mod)

1	2	3	4	5
M	$MD = DM$	$MKM = MK^*$	$K_1MDK_2 = K_1DMK_2$	X
D	MK	$DKM = KDM = KMD$	$= DK_1MK_2$	
K	~~DK~~ = DK	$DMK = MDK = MKD$	$= K_1MK_2D$	
	KM			

Stability?

$$D(z) = \frac{N(z)}{P(z)}$$

$z = 0, 5$

Does $P(z)$ have zeroes inside unitcircle?
Map $P(z)$ onto ~~[illegible]~~ $Q(s)$ so unitcircle → RHP

Could let $z = e^{-sT}$ but not rational in s.

or let $z = \frac{1-s}{1+s}$

$z=0 \to s=1$
$z=1 \to s=0$
$z=-1,\ s=\infty$
} like Smith Chart.

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$\cancel{D}$ $Q(s) = P\left(\frac{1-s}{1+s}\right)$

Does $Q(s)$ have any zeros in RHP.

Use Hurwicz, Nyquist, etc.

~~zero~~ RHP

eg: $P(z) = \left(z + \frac{1}{2}\right)(z+2)$

$$Q(s) = \left(\frac{1-s}{1+s} + \frac{1}{2}\right)\left(\frac{1-s}{1+s} + 2\right) = \frac{\frac{1}{2}(3-s)(3+s)}{1+s}$$

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Quantization

$f(t) \longrightarrow \boxed{M} \longrightarrow f^*(t) = f(t)\,c(t)$

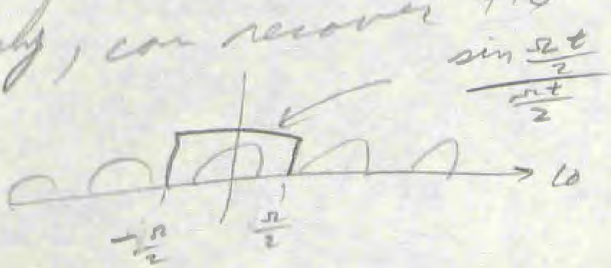
$\uparrow c(t)_{(T,\Omega)}$

$$c(t) = \frac{1}{T}\sum_{k=-\infty}^{\infty} e^{jk\Omega t}$$

$$f(t) = \frac{1}{2\pi j}\int F(s)e^{st}\,ds$$

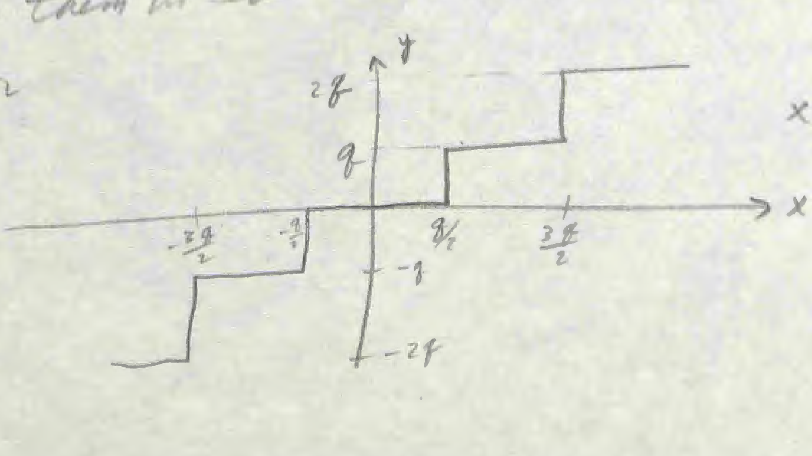
$$f^*(t) \leftrightarrow F^*(s) = \sum_{-\infty}^{\infty} F(s+jk\Omega)$$

If $F(s)$ bd ltd properly, can recover $f(t)$ from $f^*(t)$ by ideal filter

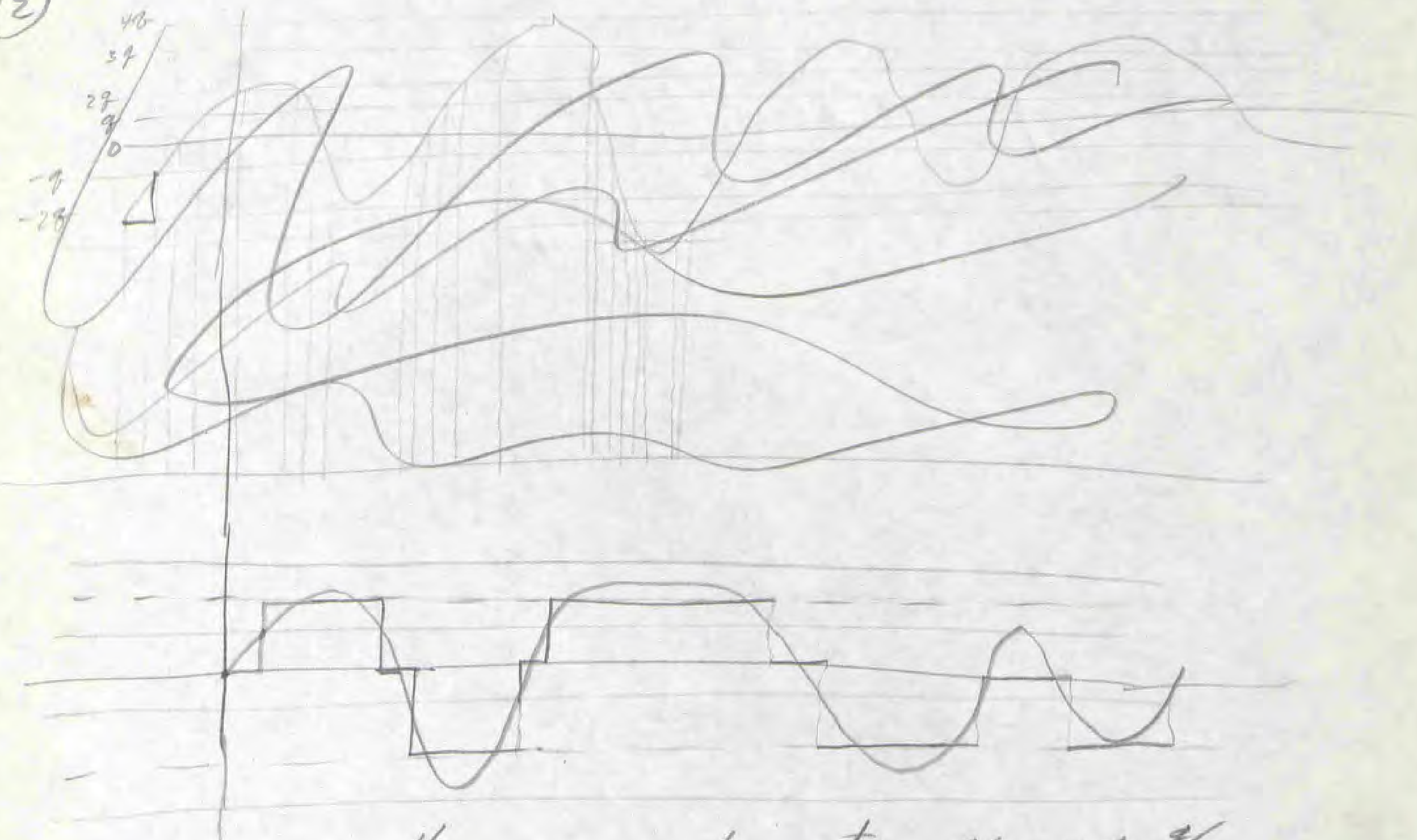


Quantize a sample signal (which is really throwing away info) because it is easier & cheaper to build systs for them in some cases.

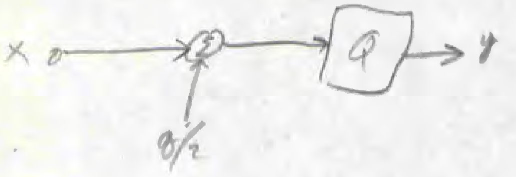
Quantizer

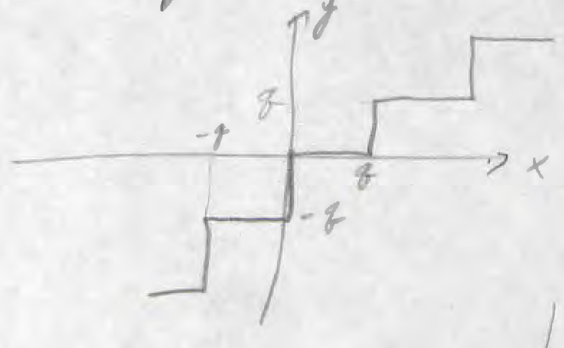


$x \longrightarrow \boxed{Q} \longrightarrow y$

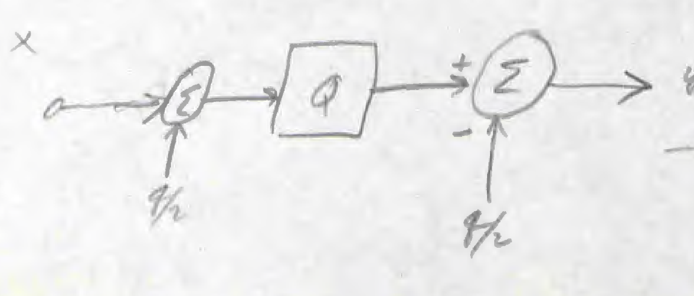


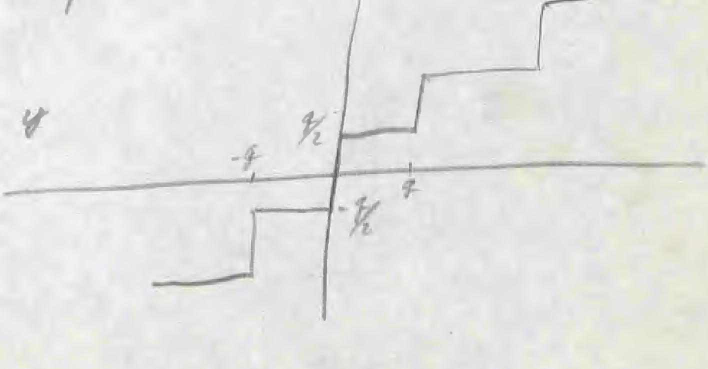
Note for a small ~~[illegible]~~ range of input $-q/2 < x < q/2$ the output is zero. This may not be desirable.





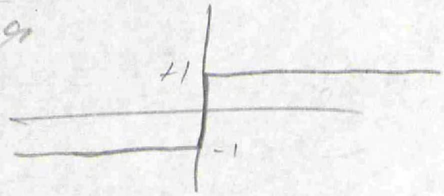
or



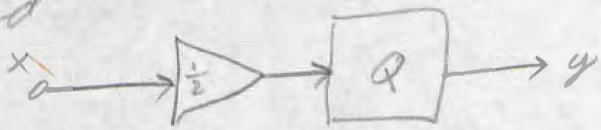


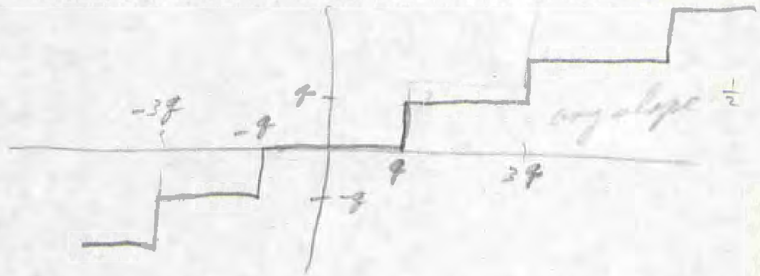
(3)

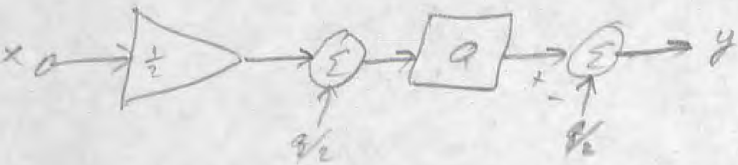
Might want to consid quantizer

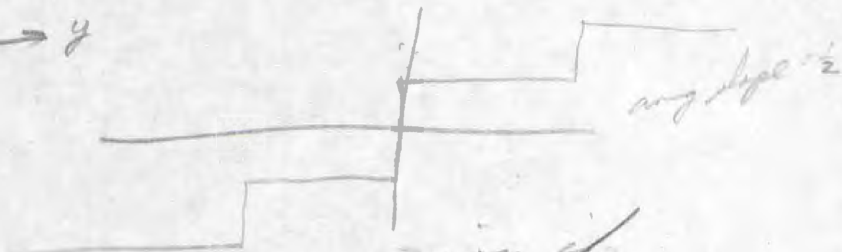


Consid

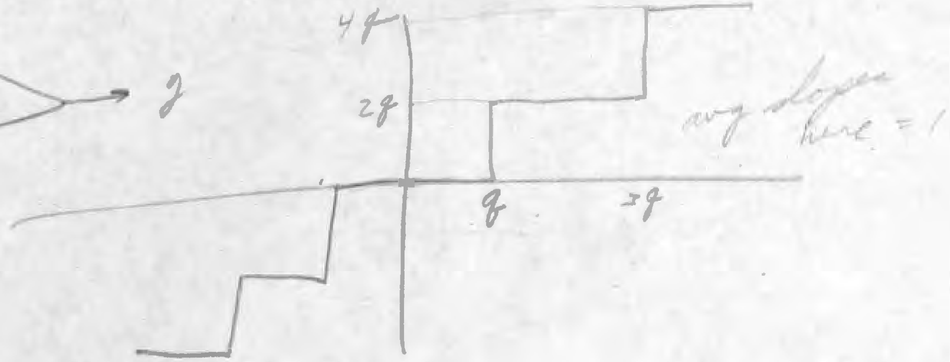








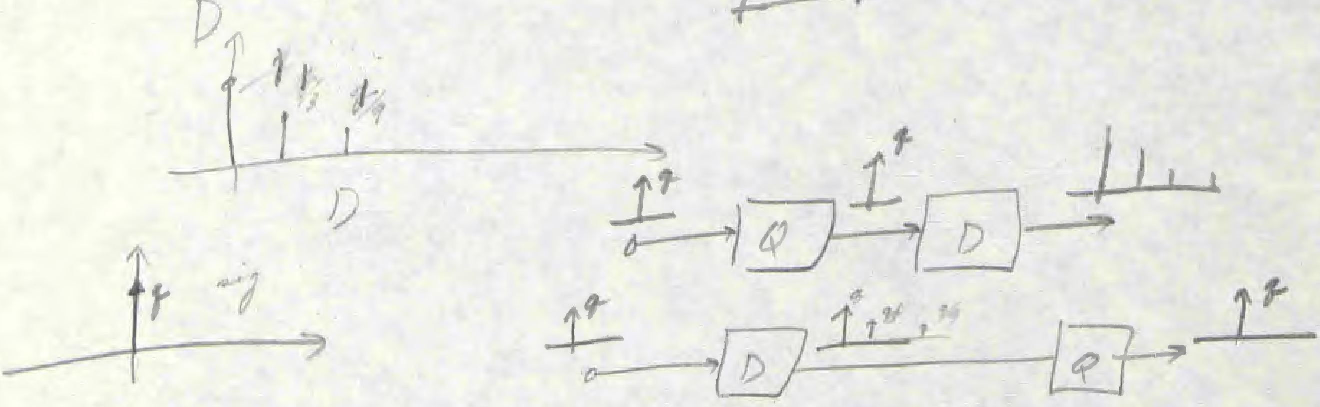
Just cutting down input to fall in lesser region of quantizer above. cut down to suff low level, get 2 valued quantizer



(8)

$QD \stackrel{?}{\equiv} DQ$

$\boxed{NO}$

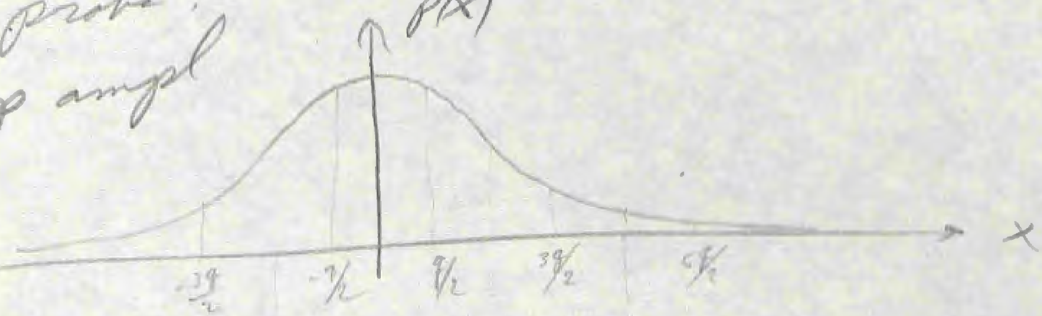


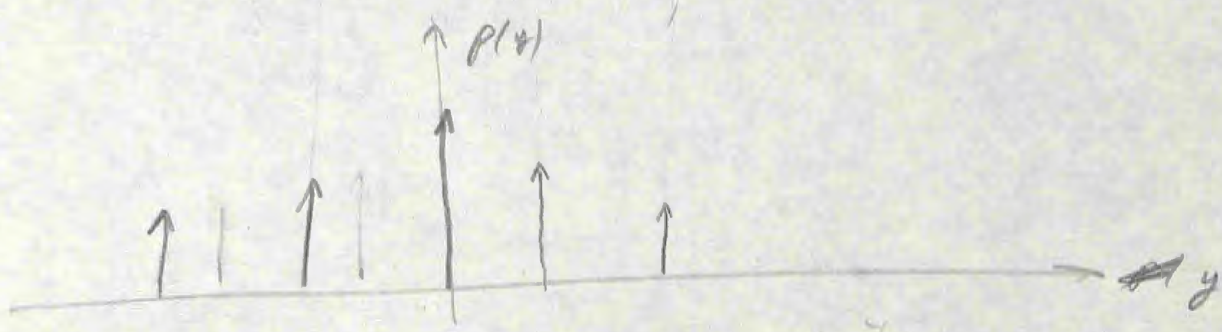
But $MQ \equiv QM$

Can't compl restruct ~~orig~~ orig sig.

Look at prob:

Prob of orig ampl





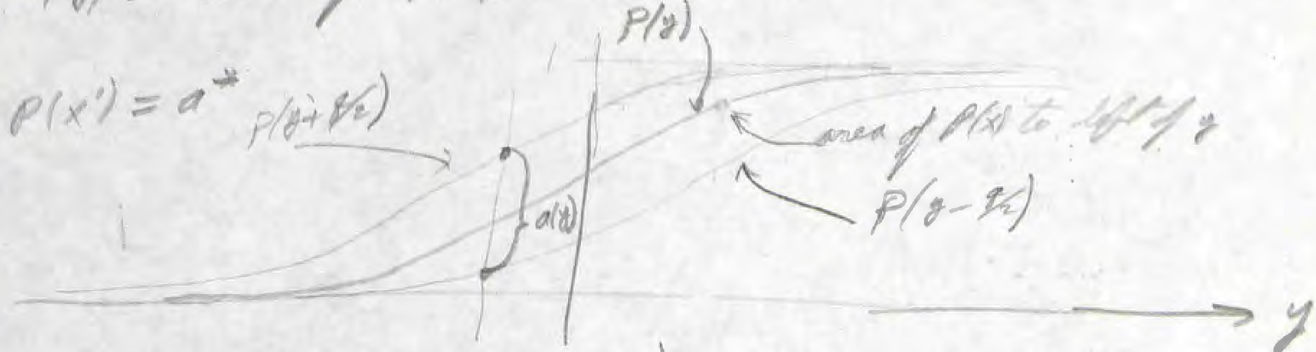
Can we convert this "area sampling" into reg sampl of sig itself?

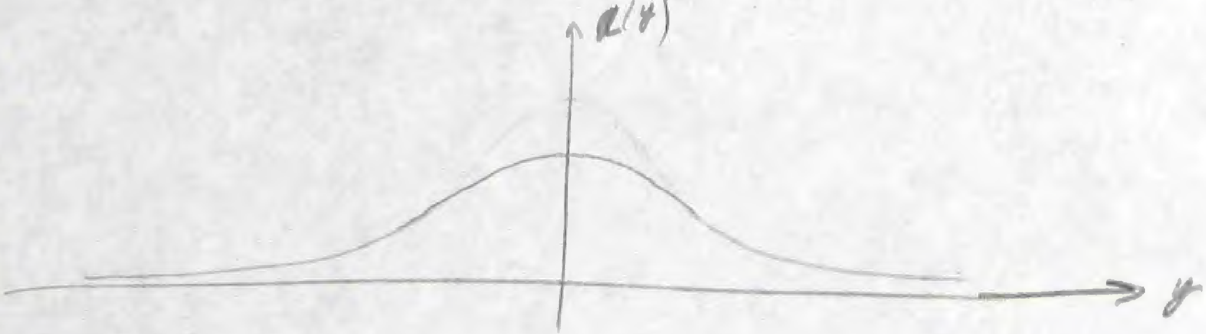
(5)

switch notation d/y → x'

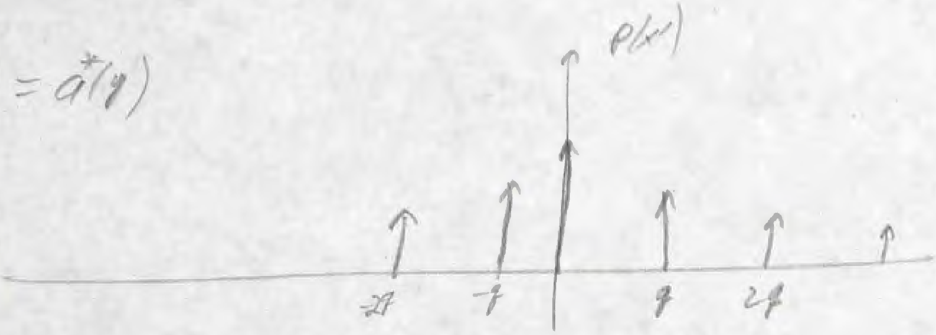
Let $a(y)$ = area of $P(x)$ betw $y - q/2$ & $y + q/2$

$P(x') = a^*$





$P(x') = a^*(y)$



Let $W_x(u) = \int_{-\infty}^{\infty} P(x) e^{jux}\,dx$

What is char fn for $a(y)$?
$P(y)$

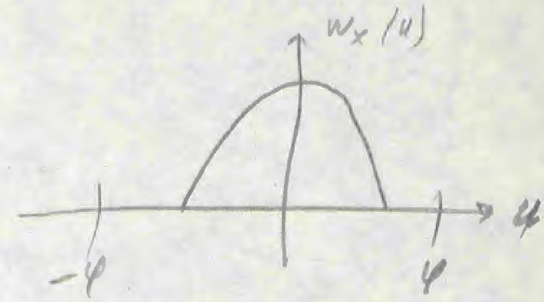
$$\left.\begin{aligned} W_y(u) &= \frac{1}{ju} W_x(u) \\ W_{y-\frac{q}{2}}(u) &= \frac{e^{-ju q/2}}{ju} W_x(u) \\ W_{y+\frac{q}{2}}(u) &= \frac{e^{juq/2}}{ju} W_x(u) \end{aligned}\right\}$$

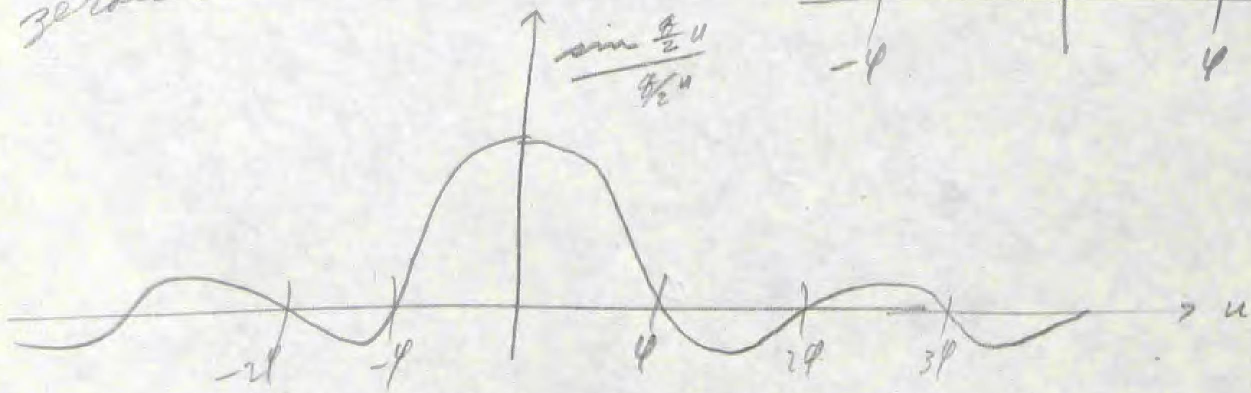
$$W_a(u) = W_x(u)\,\frac{e^{j\frac{q}{2}u} - e^{-j\frac{q}{2}u}}{ju}$$

$$= W_x(u)\, q\,\frac{\sin \frac{q}{2}u}{\frac{q}{2}u}$$

Let $\varphi = \frac{2\pi}{q}$

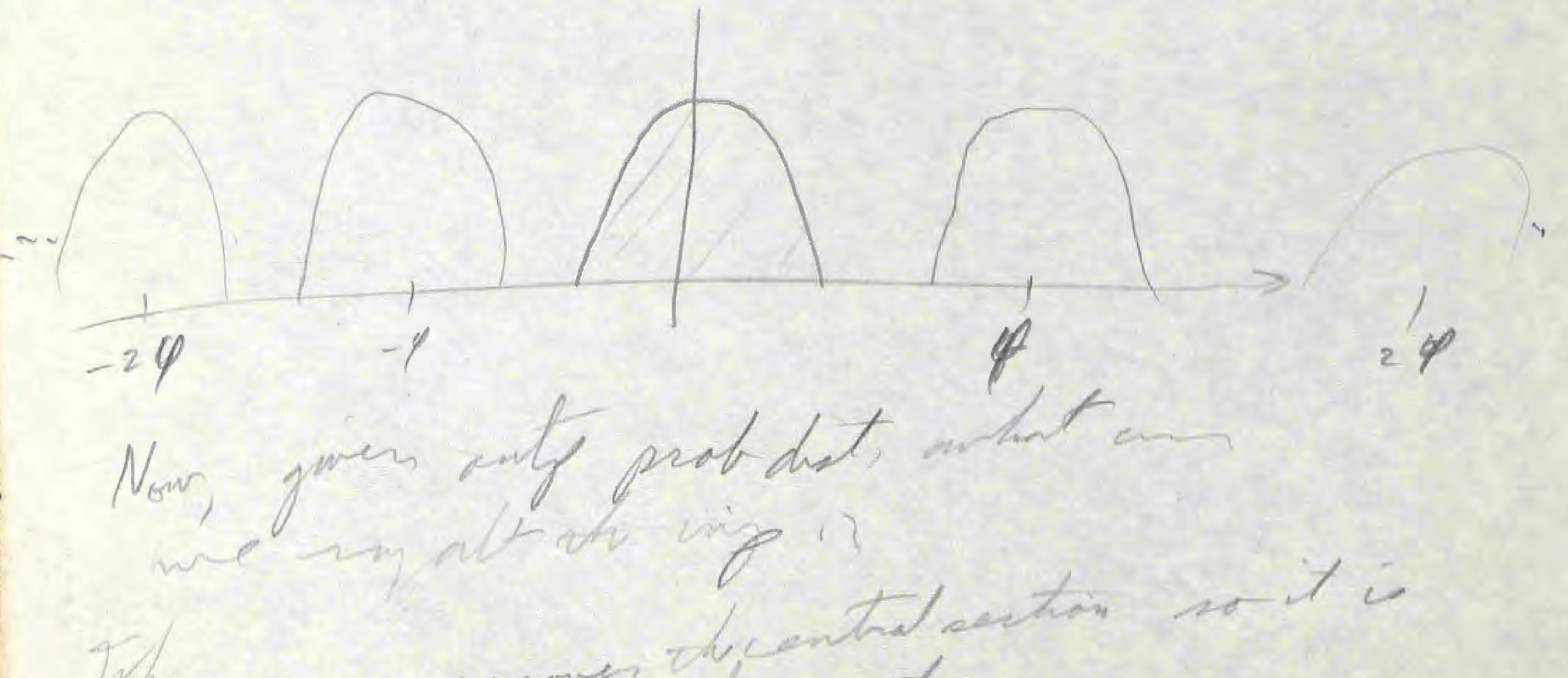
zeroes at $u = k\varphi$





$$\mathcal{F}\{\rho(x')\} = \left[W_x(u)\, q\, \frac{\sin\frac{qu}{2}}{\frac{qu}{2}} \right]^*$$

$$\mathcal{F}\{\rho(x')\} = \frac{1}{\cancel{q}} \sum W_x(u+k\varphi)\, \cancel{q}\, \frac{\sin\frac{q}{2}(u+k\varphi)}{\frac{q}{2}(u+k\varphi)}$$



Now, given only prob dist, what can we say abt the img ??

When can we recover the central section so it is not mixed w/ the other sections.

5

Can recover so long as $\quad \psi \equiv$ quantization sampling freq.

$\psi > 2[U_{max}$ of $W_x(u)]$

Now will do this:

Quantization theorem; Can recover prob fn of inp from prob den fn of outp so long as

Band ltd Char fn $\leftrightarrow$ spread out prob dens $\leftrightarrow$ spread over many quantization levels $\leftrightarrow$ should be able to do good.

$\leftrightarrow$ large dynamic range re: level size.

Suppose $z = x + n$, $\quad x$ & n indep

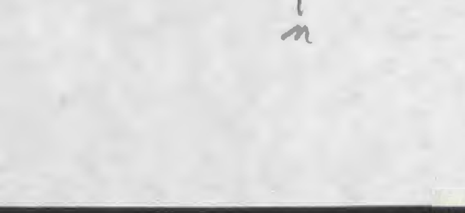
$W_z(u) = W_x(u)\, W_n(u)$

Now Central part of $\mathcal{F}\{P_1(x')\}$ is product of two fns $W_x(u)$ & $q \dfrac{\sin \frac{qu}{2}}{\frac{qu}{2}}$ so outp is Σ of 2 r.v.'s : inp & something yet to be found!

$[\quad]^*$ is ind of other parts of $[\quad]^*$

$\overline{x^k} = j^k \dfrac{\partial^k W_x(u)}{\partial u^k}\Big|_{u=0}$

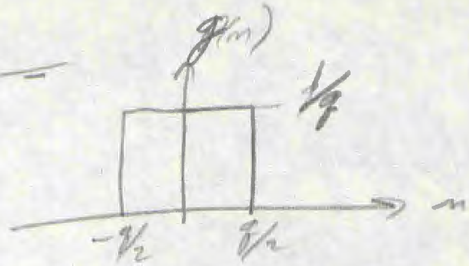
if quant theorem is sat.



x' & z have same moments

(8)

Now ~~$q(n) = \frac{1}{q}$~~ , -



$$W_n(u) = \int_{-\infty}^{\infty} q(n)\, e^{-jun}\, dn = \frac{\sin \frac{q}{2}u}{\frac{q}{2}u}$$

By treating noise as ind of input, get rep of sig
good enough to find moments of x
This is true also if quant theorem is "½ satisfied"
i.e., they can overlap ~~so long as~~ they don't overlap.

~~$x(t) \cdot s(t) \cdot m(t)$~~

~~$\int_0^T x(t)\,s(t)\,dt \gtrless \gamma$~~

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May 2

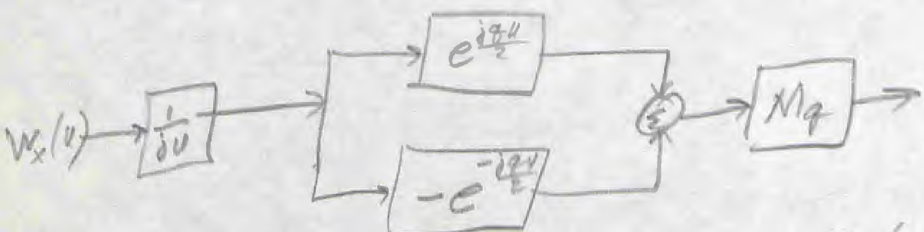
~~VI·LIV~~

VI·LIV

$x \longrightarrow \boxed{Q} \longrightarrow x'$

$W_x(u)$

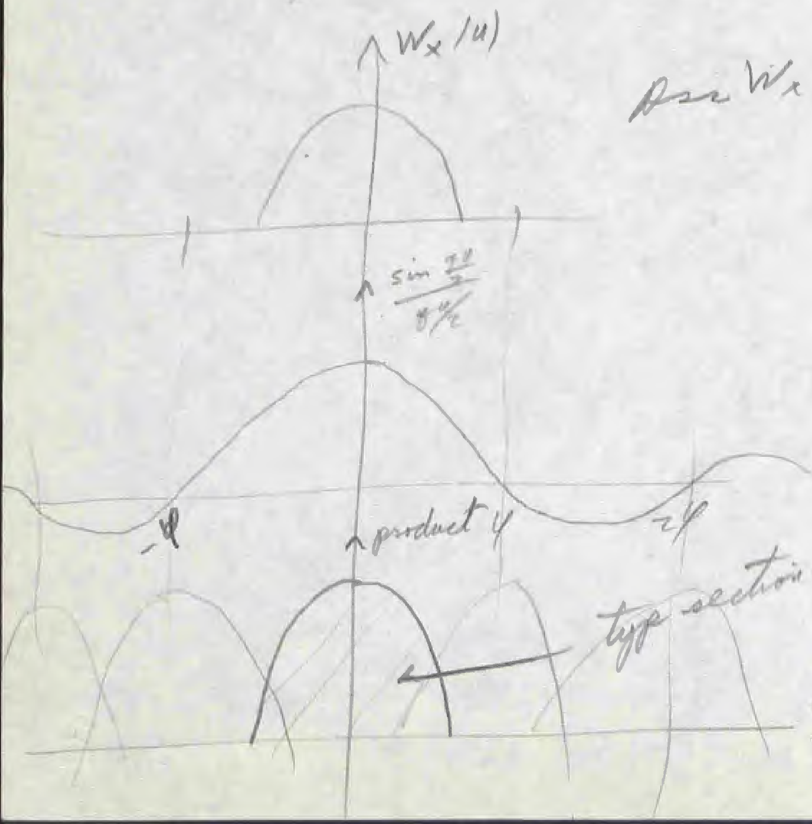
$$W_{x'}(u) = \left[W_x(u)\,\frac{e^{j\frac{qu}{2}}-e^{-j\frac{qu}{2}}}{ju}\right]^* = \left[W_x(u)\,q\,\frac{\sin\frac{qu}{2}}{\frac{qu}{2}}\right]^*$$



$\longrightarrow W_{x'}(u) = q\,\frac{\sin\frac{qu}{2}}{\frac{qu}{2}}$

$$W_{x'}(u) = \frac{1}{\cancel{q}}\sum_{k=-\infty}^{\infty} W_x(u+k\varphi)\,\cancel{q}\,\frac{\sin\frac{q}{2}(u+k\varphi)}{\frac{q}{2}(u+k\varphi)} \quad ; \quad \boxed{\varphi = \frac{2\pi}{q}}$$

Ass $W_x(u)$ band ltd



typ section of $W_x'(u)$

No overlap for proper bd lty
draw bd ltd.

(2)

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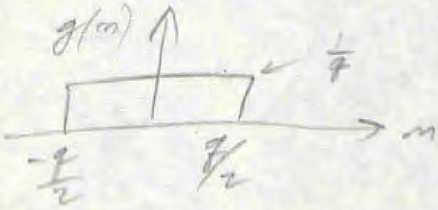
~~If $x'(t) = n(t) + x(t)$~~



$$W_n(u) = \int_{-\infty}^{\infty} g(n) e^{-jun}\, dn$$

$$= \int_{-q/2}^{q/2} \frac{1}{q} e^{-jun}\, dn$$

$$= \frac{\sin \frac{q}{2} u}{\frac{q}{2} u}$$

Since $W_{x'}(u) = W_x(u)\, W_n(u)$ for central typical section

can consid $x'(t) = n(t) + x(t)$.

Get same moments of $P_{x'}$ by eval derivs of $W_{x'}(u)$ at $u=0$; So need req that sections do not overlap to $u=0$; i.e., $W_x(u)$ ltd to $-q < u < q$. But this is just finding a contin dist w/same moments.

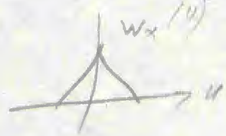
$$\overline{n^k} = j^k \left. \frac{\partial^k W_n(u)}{\partial u^k} \right|_{u=0}$$

(3)

$$\overline{x'} = \overline{x} + \overline{n} = \overline{x}$$

$$\overline{n} = 0$$

$$\overline{n^k} = \int g(n)\, n^k\, dn = \frac{1}{q}\left(\frac{q}{2}\right)^{k+1}\left[\frac{1-(-1)^{k+1}}{k+1}\right] = \begin{cases} 0, & k \text{ odd} \\ \left(\frac{q}{2}\right)^k \frac{1}{k+1}, & k \text{ even} \end{cases}$$

Ex of bd ltd prob dist is $\left(\frac{\sin x}{x}\right)^2 \leftrightarrow$ 

$$\overline{x'} = \overline{x} + \overline{n} = \overline{x}$$

$$\overline{(x')^2} = \overline{(x+n)^2} = \overline{x^2} + 2\overline{xn} + \overline{n^2} = \overline{x^2} + 2\bar{x}\overset{0}{\overline{n}} + \overline{n^2}$$

$$= \overline{x^2} + \overline{n^2} = \overline{x^2} + \frac{q^2}{12}$$

$$\cancel{\overline{(x')^3} = \overline{(x+n)^3} =}$$

or $\overline{x^2} = \overline{(x')^2} - \frac{q^2}{12}$ Sheppard's correction for grouping data.

$$\overline{(x')^3} = \overline{(x+n)^3} = \overline{x^3} + 3\overline{x^2}\,\cancel{\overline{n}} + 3\bar{x}\,\overline{n^2} + \overline{n^3}$$

$$= \overline{x^3} + \bar{x}\,\frac{q^2}{4}$$

$$\cancel{\overline{(x')^3} - 3\left(\frac{q^2}{12}\right)x'}$$

or $\overline{x^3} = \overline{(x')^3} - \frac{q^2}{4}\overline{x'}$ since $\overline{x} = \overline{x'}$

$$\overline{(x')^4} = \overline{(x+n)^4} = \overline{x^4} + 6\left(\overline{x'^2} - \frac{q^2}{12}\right)\frac{q^2}{12} + \frac{q^4}{80}$$

$12^2 = \frac{1}{144}$

or $\overline{x^4} = \overline{(x')^4} - \left(\frac{q^2}{2}\right)\overline{(x')^2} + \frac{7}{240}q^4$

$$\cancel{= \overline{(x')^4} - 6\left(\frac{q^2}{12}\right)\overline{(x')^2} + \left(\frac{q^2}{12}\right)^2}$$

(4)

$\cancel{(x-m)^4 = (x-m)(x^3 + 3x^2m + 3xm^2 + m^3)}$

$\cancel{= x^4 + 3x^3m + 3x^2m^2 + xm^3}$

$\cancel{+ \quad mx^3 + 3x^2m^2}$

$\cancel{= \overline{x^4} + 4\overline{m}\,\overline{x^3} + 6\overline{x^2}\,\overline{m^2} + 4\overline{m^3}\,\overline{x} + \overline{m^4}}$

$\cancel{= \overline{x^4} + 6\overline{x^2}\,\overline{m^2} + \overline{m^4}}$

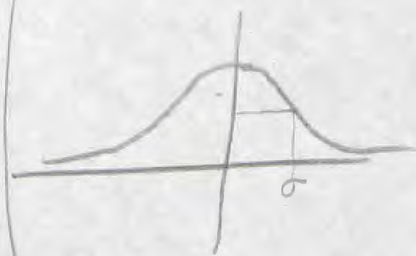
$\cancel{= \overline{x^4} + 6\overline{x^2}A +}$

12
9
14
5
40

~~Gauss~~:

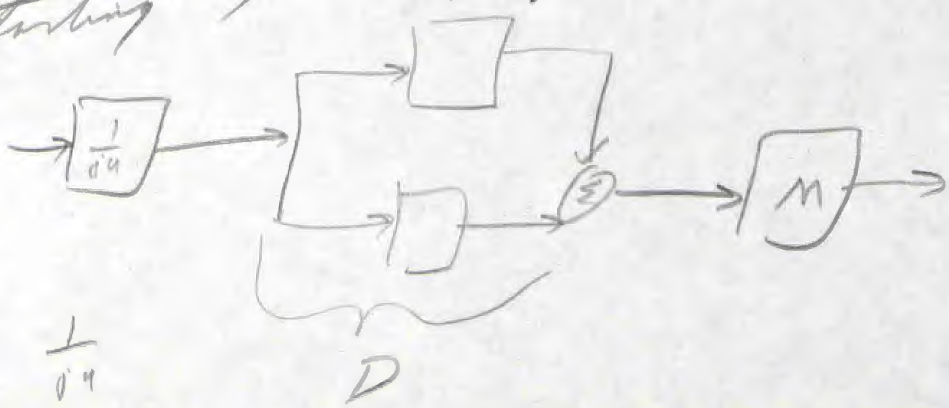
$\cancel{p(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-m)^2}{2\sigma^2}}}$

$\cancel{W_x(u) = e^{-\frac{\sigma^2 u^2}{2} - ium}}$



q	$\overline{x^2}$ meas to > actual by (%)
σ	10^{-6} %
2σ	2.3 %
3σ	31 %

Starting w/ char fn of outp, how get char fn of inp?

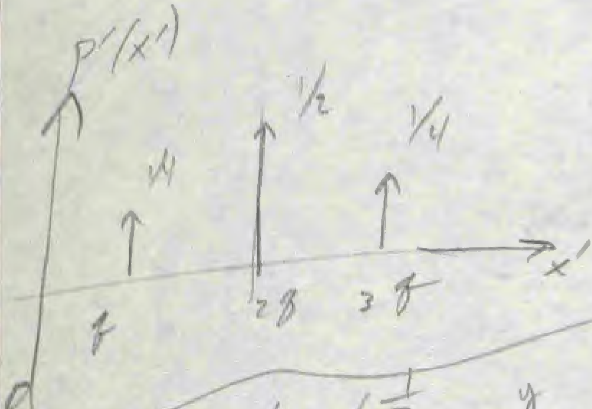


5

$$W_x(u) \to \boxed{\frac{1}{iu}} \to \boxed{D} \to \boxed{M} \to \;\equiv\; \to \boxed{\frac{1}{iu}} \to \boxed{M} \to \boxed{D} \to$$

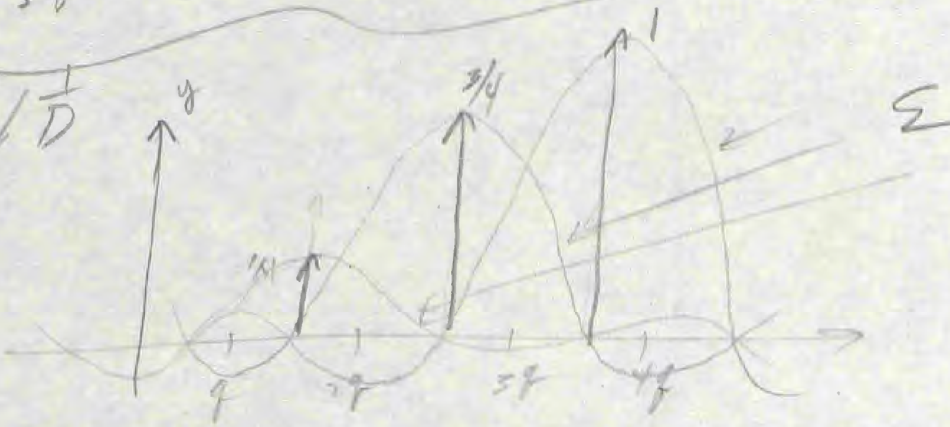
$$W_{x'}(u) \to \boxed{\frac{1}{D}} \xrightarrow{y} \boxed{H(\omega)} \xrightarrow{z} \boxed{iu} \to W_x(u)$$

$$D = \frac{1-e^{-iqu}}{e^{-i\frac{qu}{2}}} \Rightarrow \frac{1}{D} = \frac{e^{-i\frac{qu}{2}}}{1-e^{-iqu}}$$

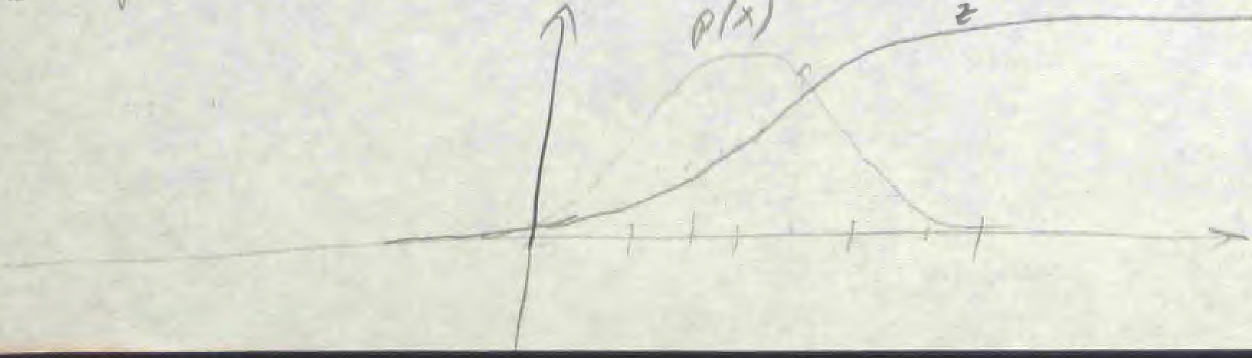


what is $p(x)$?

put through $\frac{1}{D}$



put thru low pass filter $\Rightarrow \frac{\sin x}{x}$ interpolation



Density of quant noise is
indeed flat top if quant this is real

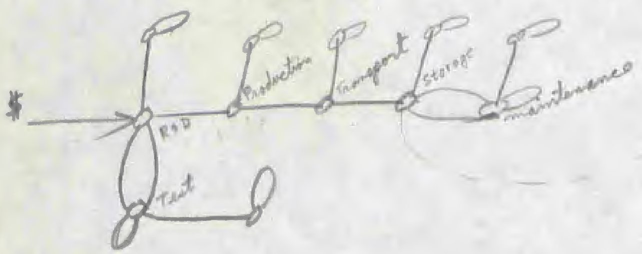
9 May

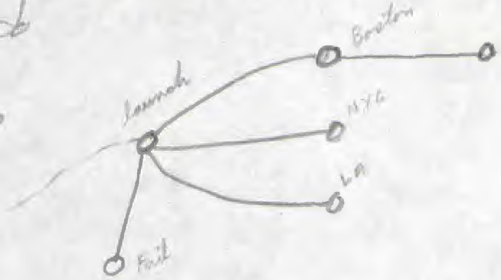
6-5-4

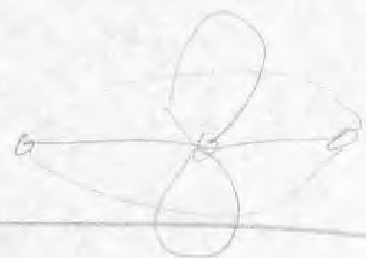
8 10

$P_x = \frac{\pi}{4}$

Missile production & maint.







~~Suppose $\{x_n\}$ characterized $\{m_n\}$ $x(t)$ $m(t)$~~

$m \rightarrow \boxed{\text{System}} \rightarrow x$

Desire x to follow "trajectory" T

$\left.\begin{array}{l}\text{discrete}\\ \text{contin}\end{array}\right\}$ time $\left\{\begin{array}{lll} m_n, & x_n, & T_n \\ m(t), & x(t), & T(t)\end{array}\right.$

Pick error measure, at each time $E(T_n, x_n)$

Σ error over period is $\sum_0^N E(\cancel{T_n}, x_n) = J(\text{initial cond}^x, m_0, m_1, \dots, m_N)$

or $\int_0^T E[T(t), x(t)]\,dt = J[\text{initial cond}^x, m(t)]$

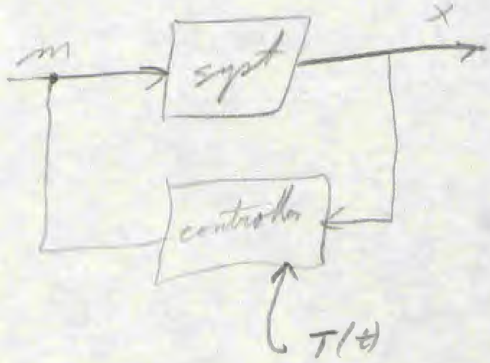
e.g. $E[T(t), x(t)] = [T(t) - x(t)]^2$

2 2 2

K0060

Obv, try to $\min_m J$

Here opt inp, not syst.



Class approach is to use calc of var

e.g. get Wiener ~~Hopf~~ eq.

Here use dynamic programming:

e.g. Terminal control problem: make x_N as close to zero as possible, given existing initial condx at $t=0$. Constrain inp "energy" $\sum_0^N m_k^2$; do this by finding

$$\min_{m_k} \left\{ \lambda \sum_0^N m_k^2 + x_N^2 \right\}$$

→ λ is a param to be decided later. It is a "policy decision"

→ or can constrain $\sum m_k^2 \le A$; then can pick λ to satisfy the constraint

→ or can pick λ beforehand (or "trade-off ratio") & determine the opt answer consist w/ this trade-off.

(3)

A K^{th} order discrete linear system is characterized by K state variables

$$x_{n+1} = \sum_{0}^{K-1} \alpha_j x_{n-j} + m_n$$

$$\alpha_0 x_n + \alpha_1 x_{n-1} + \cdots + \alpha_{K-1} x_{n-K+1}$$

recall $\underline{x}_{n+1} + A\underline{x}_n + B m_n$

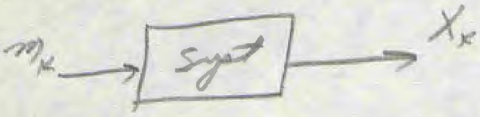
e.g. $x_N = (\beta_1 x_0^1 + \beta_2 x_0^2 + \cdots \beta_K x_0^K) + \sum_{1}^{N} H_{N-K} m_K$

11 May 61

6·54

OPB

K ØBGD



Init condx at $k=0$ are $\underline{X_0}$

Want $\left\{X_N^2 + \lambda \sum_{k=0}^{N-1} m_k^2\right\}$

to be as small as poss (time $k=N$)

So prob is to find $\min_{\{m_k\}} \left\{X_N^2 + \lambda \sum_{k=0}^{N-1} m_k^2\right\}$

Now need $X_N(\underline{m_k})$

$X_N =$ (what X_N would be if $m_k=0$; $k=0,1,\dots,N-1$)

$+ \sum_{j=0}^{N-1} H_{N-j}\, m_j$

(superscript ↔ comps of X) !

$$x_N = \underbrace{\left(\alpha^1(N)\,x_0^1 + \alpha^2(N)\,x_0^2 + \dots + \alpha^l(N)\,x_0^l\right)}_{(B_N)} + \sum_{0}^{N-1} H_{N-j}\, m_j$$

$B_k \equiv$ (what x_{j+k} would be if $m_l = 0$; $l = j, j+1, \dots, j+k-1$)

Now $x_N = B_N + \sum_{0}^{N-1} H_{N-j}\, m_j$

$$F^N(B_N) = \min_{\{m_k\}} \left[x_N^2 + \lambda \sum_{k=0}^{N-1} m_k^2 \right]$$

(2)

Supp $N=1$

$$F^1(B_1) = \min_{m_0}\left[(B_1 + H_1 m_0)^2 + \lambda m_0^2\right]$$

$$\cancel{\frac{\partial F}{\partial m_0}} \quad 0 = 2(B_1 + H_1 m_0)H_1 + 2\lambda m_0$$

$$m_0 = \frac{-B_1 H_1}{\lambda + H_1^2}$$

$$F^1(B_1) = \left(B_1 - \frac{B_1 H_1^2}{\lambda + H_1^2}\right)^2 + \frac{\lambda B_1^2 H_1^2}{(\lambda + H_1^2)^2} = \cancel{B_1^2}\,[$$

$$= \frac{B_1^2}{(\lambda+H_1^2)^2}\left[\underbrace{(\lambda + H_1^2 - H_1^2)^2}_{\lambda^2} + \lambda H_1^2\right]$$

$$= \frac{B_1^2}{(\lambda+H_1^2)^2}\left[\lambda^2 + \lambda H_1^2\right] = \frac{\lambda B_1^2}{\lambda + H_1^2}$$

$\lambda = 0$, can make $F_1 = 0$.

For N stage process, get N simultaneous eqs ~~so hr~~ diff to solve; try new approach:

"cost" is $\lambda m_0^2 + F^{N-1}(B_{N-1})$ if apply opt policy for all stages after the first.

So best can do on N stage process is to find

$$\cancel{\min_{m_0}} \quad \min_{m_0}\left\{\lambda m_0^2 + F^{N-1}(B_{N-1})\right\} = F^N(B_N)$$

(3)

Now $B_{N-1} = B_N + m_0 H_N$

What we are doing is to assume we know opt sol for ~~for N stage~~ N-1 stage process. ~~Then best we can do for~~ ~~is to find best we can do for~~ we apply some arb force at $t=0$ & then follow opt policy from there on. Want ~~to~~ min over the value of this arb force ∴

$$F^N(B_N) = \min_{m_0} \{ \lambda m_0^2 + F^{N-1}(B$$

$$\text{i.e. } F_N(B_N) = \min_{m's} \{ \underbrace{x_N^2 + \lambda \sum_1^{N-1} m_k^2}_{F^{N-1}(B_{N-1})} + \lambda m_0^2 \}$$

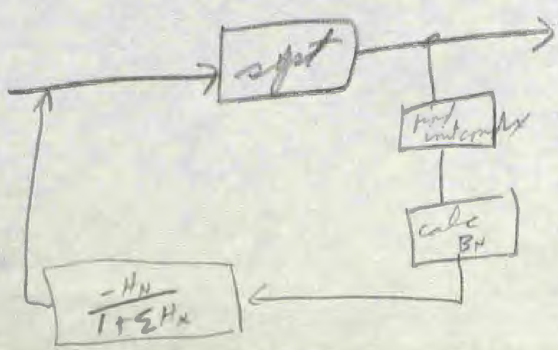
$$F^1(B_1) = \frac{\lambda B_1^2}{\lambda + H_1^2} \quad ; \quad m_0 = \frac{-B_1 H_1}{\lambda + H_1^2}$$

do the above iteration several times ; suspect

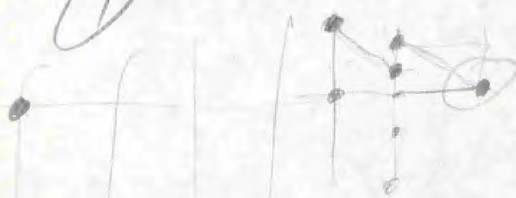
$$F^N(B_N) = \frac{\lambda B_N^2}{\lambda + \sum_1^N H_k^2} \quad ; \quad m = \frac{-B_N H_N}{\lambda + \sum_1^N H_k^2}$$

m is the opt ~~[illegible]~~ first force for N stage process

~~For~~ After the first step, have an N-1 stage process.



(4)



$F(B_1)$

$\lambda m^2 + F(B_1')$ $\quad B_1' = f(m)$

If have noise added to input,

$F_N(B_N) = \min E\left\{\sum \left(X_N^2 + \lambda \sum m_i^2\right\}\right.$; $\quad m = -\dfrac{(B_N + E_N)H_N}{\lambda + \sum H_k^2}$

$E_N =$ ~~expected value of~~ outp due to noise N units later.

$= \cancel{m} \; \bar{n} \sum H_k$

~~Let $D(z)$~~ ~~$D(z) \cdot A(z) =$~~

~~$D(z)A(z) + D(z)B(z^{-1}) \leftrightarrow D(z)D(z) + D(z)C(z^{-1})$~~

~~$D(z)$ only for neg time sig.~~

Let $\Phi_{vv}(z) = G(z)G(z^{-1})$

Now mult both sides of eq inside brackets

$$\left\{ \frac{G(z)G(\frac{1}{z})}{G(\frac{1}{z})} W(z) \right\}_+ = \left\{ \frac{\Phi_{vi}(z)}{G(\frac{1}{z})} \right\}_+$$

Now $\uparrow$ is all + time so

$$W(z) = \frac{1}{G(z)} \left\{ \frac{\Phi_{vi}(z)}{G(\frac{1}{z})} \right\}$$

$$w \longrightarrow \boxed{G(z)} \xrightarrow{\ v\ }$$

$$\Phi_{vv}(z) = G(z)G(\tfrac{1}{z})\,\Phi_{ww}(z)$$

~~unit impulse~~ white noise: $\Phi_{ww}(z) = 1$

(3)

$$\varphi_{ve}(k) = E\left\{v(n)\,e(n+k)\right\} = E\left\{v(n-k)\,e(n)\right\}$$

Define opt syst as one where

$$\varphi_{ve}(k) = 0\,,\quad k \geq 0$$

If inp & error are unrelated, then $\exists$ no operation on the error that will give a better measure of input. Hence, cannot do better.

$$\Rightarrow \left\{\Phi_{ve}(z)\right\}_{+} = 0$$

means take only contribution of the positive time portion.

$$\left\{\Phi_{vv}(z)W(z)\right\}_{+} = \left\{\Phi_{vi}(z)\right\}_{+}$$

is $\mathcal{Z}$ form of $\Phi_{ve}(z) = \Phi_{vi}(z) - \Phi_{vv}(z)W(z)$

for $\left\{\Phi_{ve}(z)\right\}_{+} = 0$.

~~$A(z) + B(z^{-1}) \leftrightarrow A(z) = C(z$~~

$$A(z) + B(z^{-1}) \leftrightarrow A(z) + C(z^{-1})$$

$$A(z) = a_0 + a_1 z + \cdots$$

$$B(z^{-1}) = b_1 z^{-1} + b_2 z^{-2} + \cdots$$

(2)

Now show

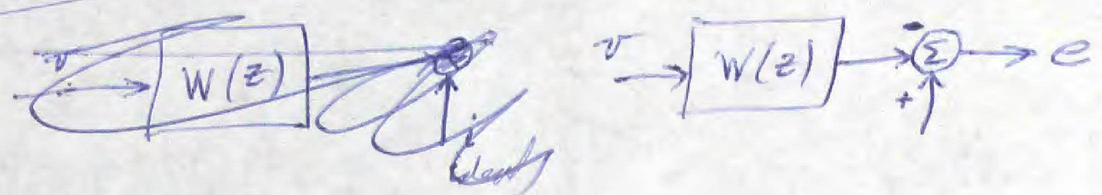
$$\Phi_{xy}(z) = \sum_i \sum_j G_i\left(\tfrac{1}{z}\right) H_j(z)\, \Phi_{x_i y_j}(z)$$

inverse xform:

$$\Phi_{xy}(z) = \sum_i \sum_j \left(\sum_{p=0}^{\infty} g_i(p) z^{-p}\right)\left(\sum_{r=0}^{\infty} h_j(r) z^{r}\right)\left(\sum_{q=\cancel{\infty}}^{\infty} \varphi_{x_i y_j}(q) z^{q}\right)$$

$$= \sum_i \sum_j \sum_{p=0}^{\infty} g_i(p) \sum_{r=0}^{\infty} h_j(r)\, \varphi_{x_i y_j}(k+p-r)\, z^{k} \quad ?$$

$$k = q + r - p$$

Consider 

Statistics of $\underline{v}$ are known:

$$E(z) = I(z) - V(z)W(z)$$

$$\Phi_{ee}(z) = \Phi_{ii}(z) - \Phi_{iv}(z)W(z) - \Phi_{vi}(z)W\left(\tfrac{1}{z}\right) + \Phi_{vv}(z)W(z)W\left(\tfrac{1}{z}\right)$$

$$\Phi_{ve}(z) = \Phi_{vi}(z) - \Phi_{vv}(z)W(z).$$

Now switch to optimal realizable system.

(1) Correlation is only statistical measure
(this is a linearity constraint)

(2) Error at n is statistically ~~statistically~~ unrelated to all past & present values of v.

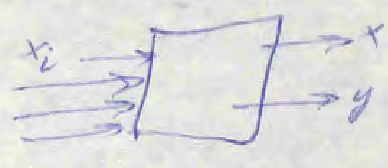
6.54 Discrete random processes

$$\varphi_{xy}(k) = E\left[x(n)\,y(n+k)\right]$$

$$\Phi_{xy}(z) = \sum_{k=-\infty}^{\infty} \varphi_{xy}(k) z^{k}$$

$$\Phi_{xx}(z) = \Phi_{xx}\left(\frac{1}{z}\right) \quad ; \quad \Phi_{xy}(z) = \Phi_{yx}\left(\frac{1}{z}\right)$$ want $\Phi_{xy}(z) = f\left(\begin{matrix}\text{input}\\\text{statistics}\end{matrix}\right)$

Let x & y be sigs at two pts of a lin syst; ~~want $\Phi_{xy}(z) = f$(input statistics)~~



$$X(z) = \sum_i G_i(z)\, X_i(z)$$ ← excitations

$$Y(z) = \sum_j H_j(z)\, Y_j(z)$$

$$x(n) = \sum_i \sum_{p=0}^{\infty} x_i(n-p)\, g_i(p)$$

$$y(n) = \sum_j \sum_{r=0}^{\infty} y_j(n-r)\, h_j(r)$$

$$\varphi_{xy}(k) = E\left[x(n)\,y(n+k)\right] = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x(n)\,y(n+k)$$

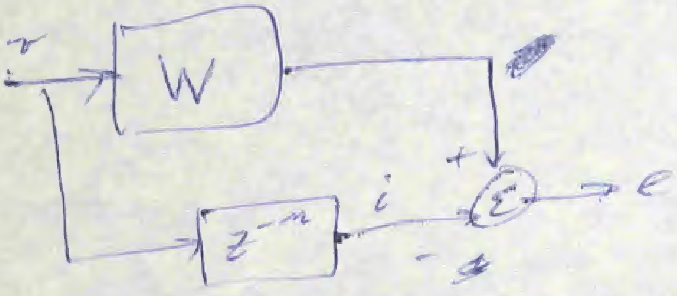
$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \left[\sum_i \sum_{p=0}^{\infty} x_i(n-p)\, g_i(p)\right]\left[\sum_j \sum_{r=0}^{\infty} y_j(n+k-r)\, h_j(r)\right]$$

$$= \sum_i \sum_j \sum_{p=0}^{\infty} g_i(p) \sum_{r=0}^{\infty} h_j(r) \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x_i(n-p)\, y_j(n-r+k)$$

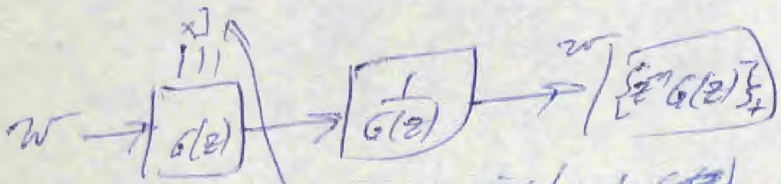
$$= \sum_i \sum_j \sum_{p=0}^{\infty} g_i(p) \sum_{r=0}^{\infty} h_j(r)\, \varphi_{x_i y_j}(p+k-r)$$

⑥

~~Predictor~~ Consider opt predictor:



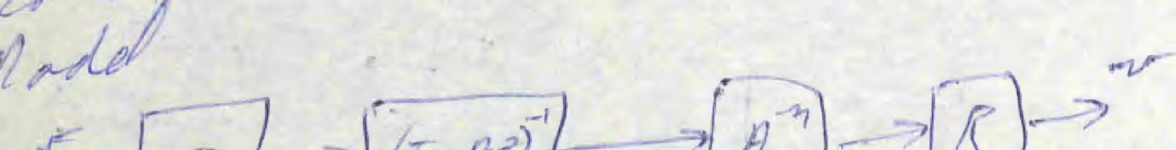
$$\text{opt } W(z) = \frac{1}{G(z)} \left\{ z^{-n} G(z) \right\}_+$$

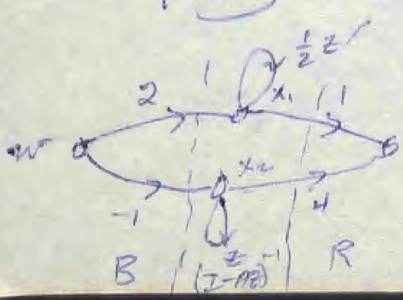


state variables of ~~G(z)~~ system G.

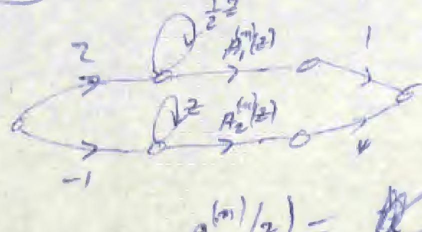
predictor takes present value of state variables, ignores all future poss sigs & computes where present ~~and~~ condx will carry system:

Model





or predictor



$$A_1^{(n)}(z) = \left(\tfrac{1}{2} z\right)^n$$

(5)

$$X(n+1)] = \underline{A}\, X_n] + \underline{B}\, w(n+1)]$$

If $w(n) = 0$, all n & know $x(0)]$

$$x(k)] = \underline{A}^{k}\, X(0)]$$

Now Z-transform

$$z^{-1} x(z)] - z^{-1} X(0)] = \underline{A}\, x(z)] + z^{-1} B\, w(z)] \quad ; \; w(0) = 0.$$

$$x(z) - x(0) = z A x(z) + B w(z)$$

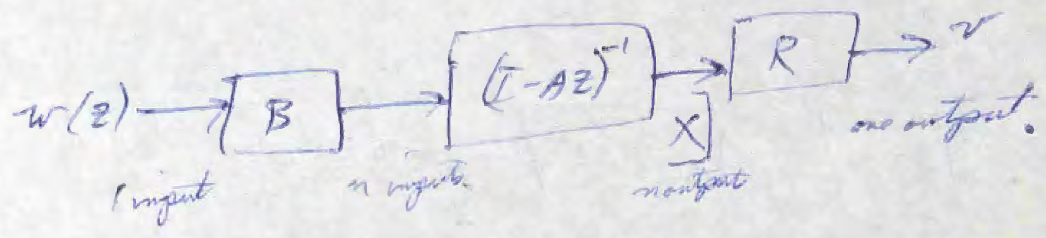
$$(I - A z)\, X(z) = X(0) + B w(z)$$

$$x(z) = (I - A z)^{-1} \left[X(0) + B w(z) \right]$$

If $w(0) = 0$, all time.

$$\cancel{x(z) =} \; [I - A z]^{-1} = \sum_{n=0}^{\infty} A^{n} z^{n}.$$

If $x(0) = 0$;



6.54 C. T. Whitehead

#1 $f(t) \leftrightarrow F(\omega)$

$A^*(t) \leftrightarrow A^*(s)$; $B^*(t) \leftrightarrow B^*(s)$

(a)

$$\boxed{A^*(\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} F(\omega - k\Omega)}$$

1. 15
2. 43
3. 7

65

$B^*(t) = f(t)\,c(t-\delta)$

impulse train

$f(t) \leftrightarrow F(\omega)$; $c(t) \leftrightarrow C(\omega)$

$c(t-\delta) \leftrightarrow e^{-j\omega\delta}\,C(\omega)$

An easy way of getting $B^*(\omega)$ is to shift $f(t)$, sample, & then shift ~~back~~ the sampled signal back:

$f(t+\delta) \leftrightarrow F(\omega)e^{j\omega\delta}$

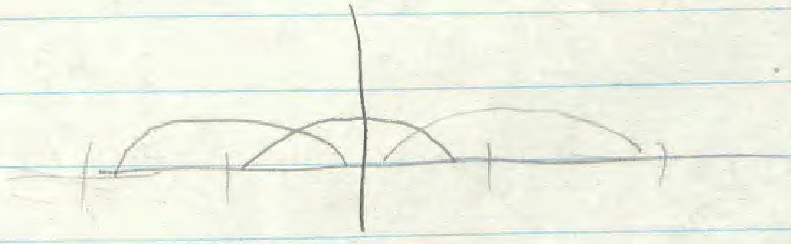
$f(t+\delta)c(t) \leftrightarrow \frac{1}{T}\sum_{k=-\infty}^{\infty} F(\omega-k\Omega)e^{j(\omega-k\Omega)\delta}$

$B^*(t) = f(t)c(t-\delta) \leftrightarrow e^{-j\omega\delta}\frac{1}{T}\sum_{k=-\infty}^{\infty} F(\omega-k\Omega)e^{j\omega\delta}e^{-jk\Omega\delta} = B^*(\omega)$

$$\boxed{B^*(\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} F(\omega-k\Omega)e^{-jk\Omega\delta}}$$

#1, (b)

~~$A^*(\omega)$ & $B^*(\omega)$ can also be written as~~



~~Put B^* through a filter that~~ multiplies

$$\boxed{T=1,\ e^{-aT}=\tfrac{1}{3}}$$

#2

(a) $F(s) = \frac{1}{s}\left[1 - z + z^2 - z^3 + \cdots\right]$

$$\boxed{F(s) = \frac{1}{s}\left[\frac{1}{1+z}\right],\quad z = e^{-sT}}$$

(b) $$\boxed{G(s) = \frac{F(s)\,a}{s+a} = \frac{a}{s(s+a)(1+z)}}$$

$$G^*(s) = \frac{1}{1+z}\left[\frac{a}{s(s+a)}\right]^* = \frac{1}{1+z}\left[\frac{1}{s} + \frac{-1}{s+a}\right]^*$$

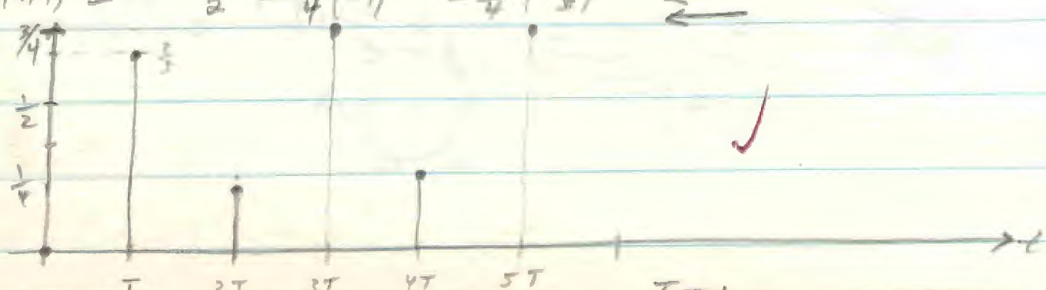
$$= \frac{1}{1+z}\left[\frac{1}{1-z} - \frac{1}{1-e^{-aT}z}\right]$$

$$= \frac{1}{1-z^2} + \frac{-1}{(1+z)(1-\frac{1}{3}z)}$$

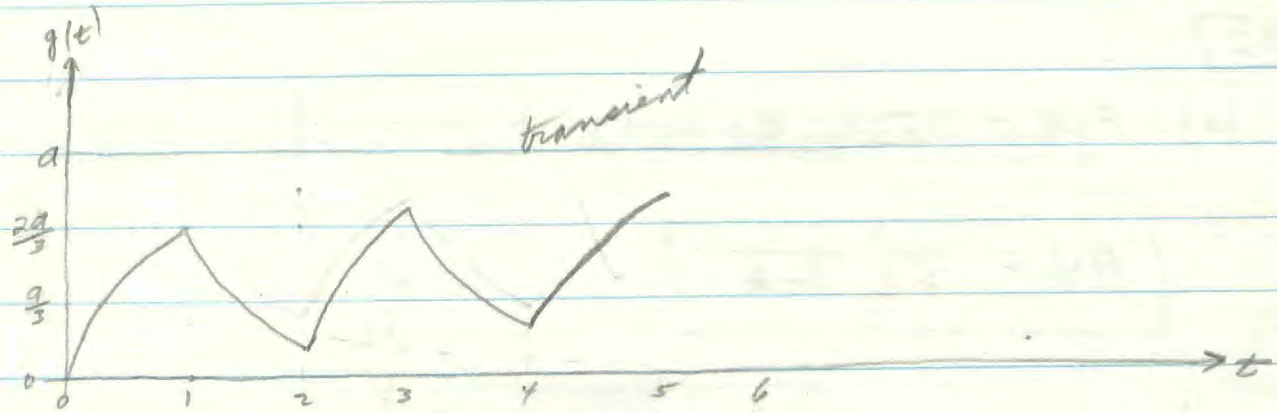
$$\boxed{G^*(s) = \underbrace{\frac{1}{1-z^2}}_{\frac{1}{(1-z)(1+z)}} + \frac{-\frac{3}{4}}{1+z} + \frac{-\frac{1}{4}}{1-\frac{1}{3}z}}$$

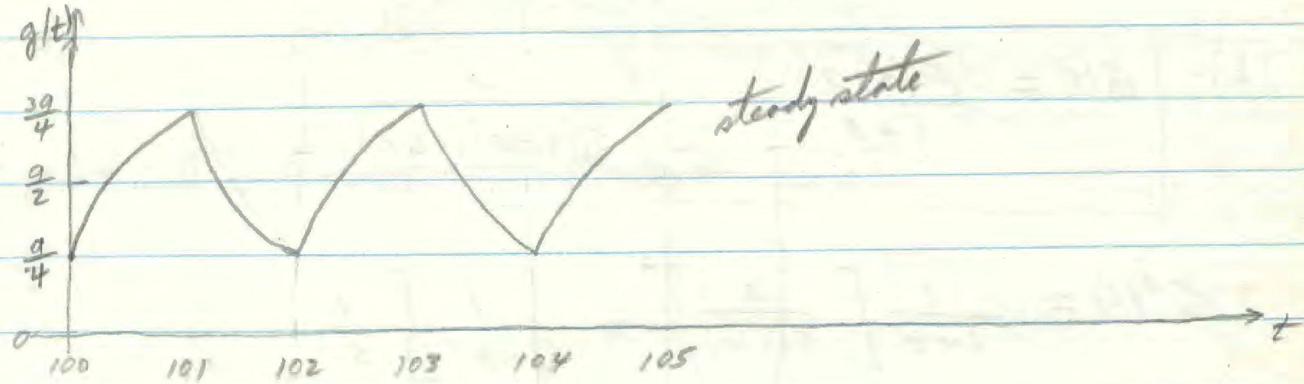
$$\boxed{G^*(s) = \frac{\frac{1}{2}}{1-z} + \frac{-\frac{1}{4}}{1+z} + \frac{-\frac{1}{4}}{1-\frac{1}{3}z}}$$

$$g^*(nT) = \frac{1}{2} - \frac{1}{4}(-1)^n - \frac{1}{4}\left(\frac{1}{3}\right)^n \Longleftarrow$$

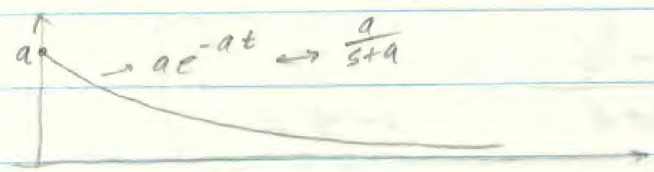


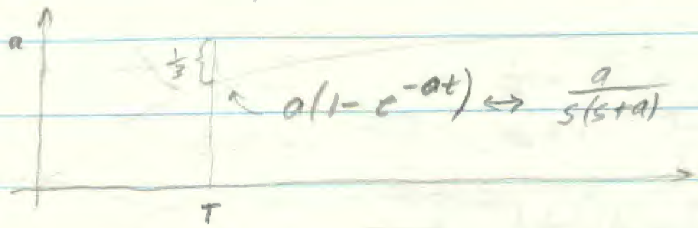
(c)





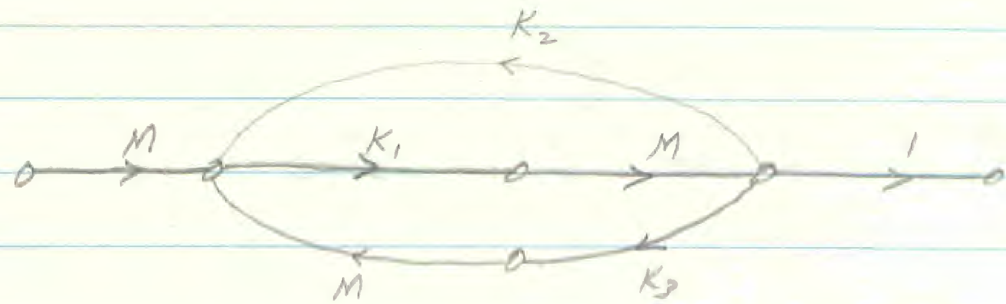
$$G(s) = \frac{a}{s(s+a)(1+z)}$$

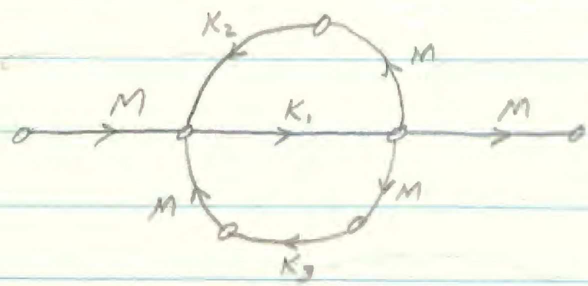


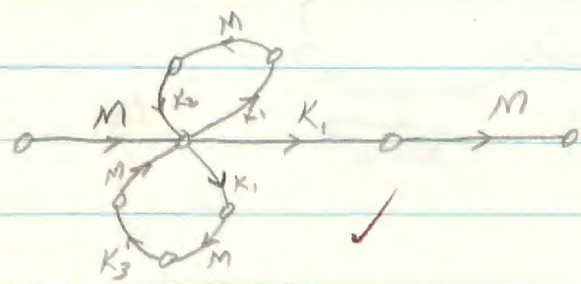


note $\frac{a}{s(s+a)} \Longleftrightarrow 1-e^{-at}$

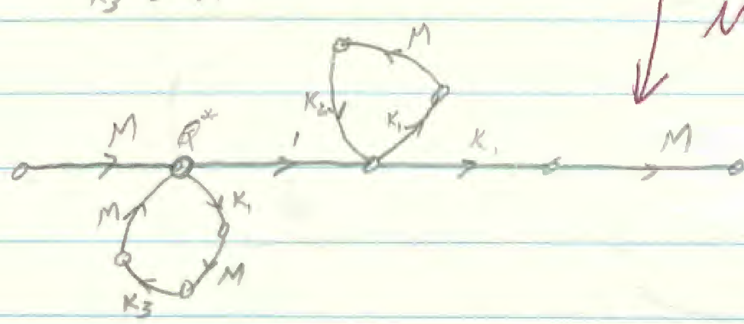
#3







not equivalent

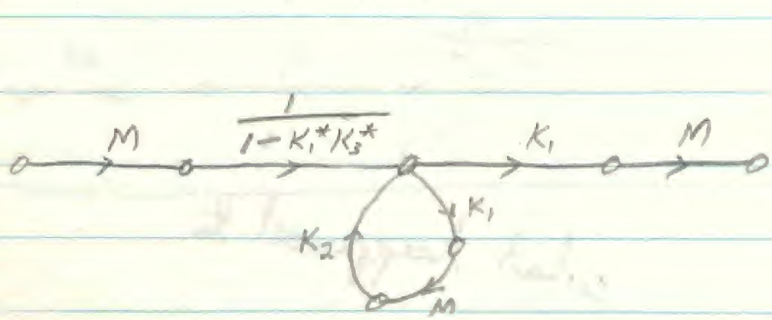


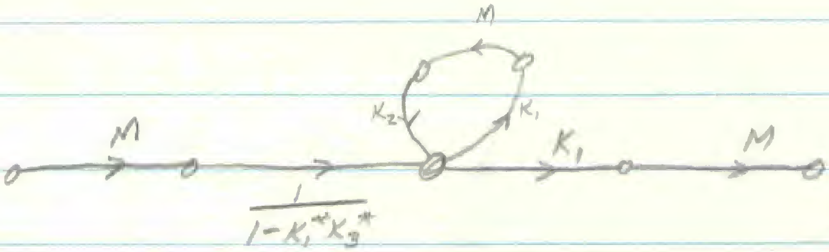
$$Q^* K_1 M K_3 M + F^* = Q^*$$

$$Q^* = \frac{F^*}{1 - K_1 M K_3 M}$$

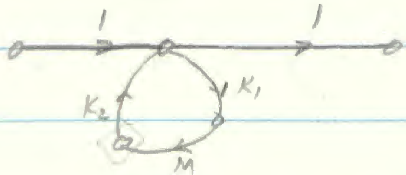
$$(Q^* K_1)^* = Q^* K_1^* = P^*$$

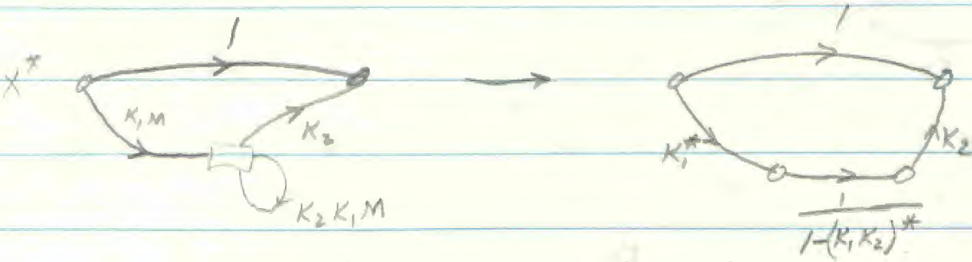
$$(P^* K_3)^* = P^* K_3^*$$



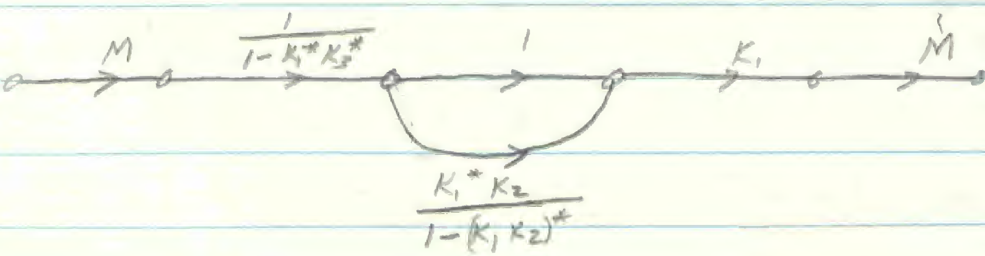


Now reduce

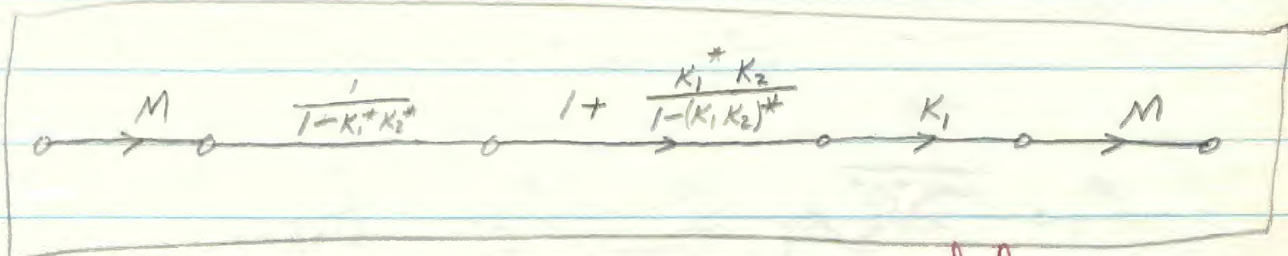




So the total flow graph is



or



$$\frac{1}{1-K_1^* K_2^*}$$

what happened to K_3

$\left. \cancel{g}_t \right]$ ~~words and column~~. row, column

$\cancel{\left[g_{t,\tau} \right] =}$

$$\left[g_{\tau,t} \right] = \begin{bmatrix} g(t_1+\tau_1) & g(t_2+\tau_1) & \cdots \\ g(t_1+\tau_2) & g(t_2+\tau_2) & \cdots \\ - & - & - \end{bmatrix}$$

$$\left. g_t \right] = \left. \begin{matrix} g(t_1) \\ g(t_2) \\ \vdots \end{matrix} \right] = \cancel{\left[A_{\tau,t} \right] \left. f_t \right]} \; \left[A_{t,\tau} \right] \left. f_\tau \right]$$

$$\left[g_{\tau,t} \right] = \left[A_{\tau,\sigma} \right] \begin{bmatrix} f(t_1+\sigma_1) & f(t_2+\sigma_1) & \cdots \\ f(t_1+\sigma_2) & f(t_2+\sigma_2) & \cdots \\ - & - & - \end{bmatrix}$$

$$= \left[A_{\tau,\sigma} \right] \left[f_{\sigma,t} \right]$$

$$\left. \varphi_{gg}(\tau) \right] = \left. \begin{matrix} \varphi_{gg}(\tau_1) \\ \varphi_{gg}(\tau_2) \\ \vdots \end{matrix} \right] = \left[g_{\tau,t} \right] \left. g_t \right]$$

$$\left. \begin{matrix} \varphi_{gg}(\tau_1) \\ \varphi_{gg}(\tau_2) \\ \vdots \end{matrix} \right] = \left[A_{\tau,\sigma} \right] \left[f_{\sigma,t} \right] \left[A_{t,\rho} \right] \left. f_\rho \right]$$

$$\left[g_{\tau,t}\right]_t = \left\{\left[A_{\tau,\sigma}\right]\left[f_{\sigma,t}\right]\right\}_t$$

$$\left[g_{t,\tau}\right] = \left[f_{t,\sigma}\right]\left[A_{\sigma,\tau}\right]$$

$$g_t] = \left[A_{t,\sigma}\right] f_\sigma]$$

$$\cancel{\text{so } \varphi_{fg_\tau}] = \left[f_{\tau,t}\right] g_t] =}$$

$$\varphi_{ff_\sigma}] = \left[f_{\sigma,t}\right] f_t]$$

$$\underline{\varphi_{ff_\sigma}} = \underline{f_t}\left[f_{t,\sigma}\right]$$

$$\varphi_{fg_\tau}] = \left[f_{\tau,t}\right] g_t]$$

so $$\underline{\varphi_{fg_\tau}} = \underline{f_t}\left[g_{t,-\tau}\right] = \underline{f_t}\left[f_{t,\sigma}\right]\left[A_{\sigma,-\tau}\right]$$

$$= \underline{\varphi_{ff_\sigma}}\left[A_{\sigma,-\tau}\right]$$

$$\underline{\varphi_{fg_\tau}} = \underline{g_t}\left[f_{t,\tau}\right]$$

$$g_t] = \left[A_{t,\sigma}\right] f_\sigma]$$

$$\underline{g_t} = \underline{f_\sigma}\left[A_{\sigma,\tau}\right]$$

$$\cancel{\varphi_{fg_\tau}] = f_{t,\tau}}$$

$$(AB)_t = B_t A_t$$

$$(ABC)_t = (AB\cdot C)_t = (DC)_t = C_t D_t$$

$$= C_t (AB)_t = C_t B_t A_t$$

$$\varphi_{fg_\tau}] = [f_{\tau,t}]\; g_t] =$$

$$\llcorner \varphi_{fg_\tau} \lrcorner = \llcorner f_t \lrcorner\, [g_{t,-\tau}] = \llcorner f_t \lrcorner\, A_{t,\sigma}\, [f_{\sigma,-\tau}]$$

$$\varphi_{fg_\tau}] = [f_{-\tau,\sigma}]\, A_{\sigma,t}\; f_t]$$

$$\cancel{E^{-1}\varphi_{fg_\tau} = E^{-1} f_{-\tau,\sigma}\, E E^{-1} A_{\sigma,t}\, E E^{-1} f_t]}$$

$$\cancel{\llcorner \varphi_{fg_\tau} \lrcorner = \llcorner g_t \lrcorner\, [f_{t,\tau}]\, A}$$

$$\cancel{g_t] = [A_{t,\sigma}]\, f_\sigma]}$$

$$\llcorner g_t \lrcorner = \llcorner f_\sigma \lrcorner\, [A_{\sigma,t}] \quad \text{from} \quad g_t] = [A_{t,\sigma}]\, f_\sigma]$$

$$\cancel{\varphi_{fg_\tau}] = [f_{\tau,t}]\, g_t] = \llcorner f_t \lrcorner\, [g_{t,-\tau}] = \llcorner f_t \lrcorner}$$

$$\cancel{\varphi_{fg_\tau} \quad [g_{t,\tau}] =}$$

$$\left(E^{-1}A_{\tau\sigma}E\right)\left(E^{-1}f_{\sigma t}E\right)\underbrace{\left(E^{-1}A_{t,\rho}E\right)E^{-1}f_{\rho}} = E^{-1}\varphi_{gg_\tau}] = \Phi_{gg_s}]$$

$$E^{-1}f_\rho = F_s \qquad \begin{bmatrix} H(s_1) & & 0 \\ & H(s_2) & \\ 0 & & \ddots \end{bmatrix}\begin{bmatrix} F(s_1) \\ F(s_2) \\ \vdots \end{bmatrix} = \begin{bmatrix} H(s_1)F(s_1) \\ H(s_2)F(s_2) \\ \vdots \end{bmatrix} = G_s]$$

$$\cancel{D F_s = \begin{bmatrix} H(s_1) & \\ & \ddots \end{bmatrix}}$$

$$E\, G_s] = g_t]$$

$$\begin{bmatrix} f(t_1+\sigma_1) & f(t_2+\sigma_1) & \\ f(t_1+\sigma_2) & f(t_2+\sigma_2) & \\ - & - & - \end{bmatrix}\begin{bmatrix} g(t_1) \\ g(t_2) \\ \vdots \end{bmatrix}$$

$$\cancel{E}\ \Phi_{gg_s}] = E^{-1}A_{\tau\sigma}E\,E^{-1}\underbrace{f_{\sigma t}\,g_t}_{\varphi_{fg}(\sigma)}$$

$$\Phi_{gg_s}] = \begin{bmatrix} H(s_1) & & 0 \\ & H(s_2) & \\ 0 & & \ddots \end{bmatrix}\Phi_{fg_s}$$

$$\Phi_{fg_s}] = E^{-1}\varphi_{fg_\tau}] = \cancel{E}$$

$$\cancel{\varphi_{fg_\tau} = f_{\sigma t}\,g_t \;;\; g_t = A_{t\sigma}f_\tau}$$

$$\cancel{\text{tat}}\ \boxed{\varphi_{fg_\tau} = f_{\tau t}\,g_t \;;\; g_t = A_{t\sigma}f_\sigma}$$

$$\varphi_{fg_\tau}] = f_{\tau t}\,A_{t\sigma}f_\sigma$$

$$E^{-1}\varphi_{fg_\tau}] = E^{-1}[f_{\tau t}]\,E\left(E^{-1}A_{t\sigma}E\right)\underbrace{E^{-1}f_\sigma}_{F_s]} = \left(E^{-1}f_{\tau t}E\right)\begin{bmatrix} H(s_1) & & 0 \\ & H(s_2) & \\ 0 & & \ddots \end{bmatrix}F_s]$$

$$\cancel{\begin{bmatrix} H(s_1)F(s_1) \\ H(s_2)F(s_2) \\ \vdots \end{bmatrix}}$$

$$\neq$$

$$\left.\varphi_{gf_\tau}\right] = g_{\tau t}\, f_t = \cancel{} = A_{\tau\sigma}\, f_{\sigma t}\, f_t$$

$$\left.\Phi_{gf_s}\right] = E^{-1} A_{\tau\sigma} E\, E^{-1} \varphi_{ff_\sigma} = \cancel{A}$$

$$= \left[\begin{matrix} H(s_1) & & \\ & H(s_2) & 0 \\ 0 & & \ddots \end{matrix}\right] \left.\Phi_{ff_s}\right]$$

$$E E^{-1} = E^{-1} E$$

$$\left.\varphi_{fg_\tau}\right] = f_{\tau t}\, g_t \rightarrow \cancel{\underbrace{\varphi_{fg_\tau}} = \Phi_t}\ f_t$$

$$\cancel{\underbrace{\varphi_{fg_\tau}} = f_t \left[g_{t\tau}\right]}$$

$$\cancel{\left.\varphi_{fg}\right] = \left[g_{\tau t}\right] f_t}$$

$$\varphi_{fg}(\tau_1) = \left.\underbrace{f_{t+\tau_1}}\ g_t\right] = \left.\underbrace{f_{t'}}\ g_{t'-\tau_1}\right]$$

$$\varphi_{fg}(\tau_2) = \left.\underbrace{f_{t+\tau_2}}\ g_t\right] = \left.\underbrace{f_{t'}}\ g_{t'-\tau_2}\right]$$

$$\cancel{\left.\varphi_{g_\tau}\right] = f}$$

$$\underbrace{\varphi_{fg_\tau}} = \underbrace{f_t} \left[g_{t,-\tau}\right]$$

$$\left.\varphi_{fg_\tau}\right] = \left[g_{-\tau,t}\right] \left. f_t\right] = \cancel{A\ g_{-\tau,t}\ f_t} = A_{-\tau,\sigma}\, f_{\sigma,t}\, f_t$$

$$\left.\Phi_{fg_s}\right] = E^{-1} \left.\varphi_{fg_\tau}\right] = E^{-1} A_{-\tau,\sigma} E\, E^{-1} \varphi_{ff_\sigma} = E^{-1} A_{-\tau,-\sigma} E\, E^{-1} \varphi_{ff_{-\sigma}}$$

$$\left.\Phi_{gg_s}\right] = \left[H(s)\right] \left.\Phi_{fg_s}\right] = \left[H(s)\right] \left[H(-s)\right] \left.\Phi_{ff_s}\right]$$