STATISTICAL THEORY OF

NON-LINEAR SYSTEMS

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STATISTICAL THEORY OF NONLINEAR SYSTEMS

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Operators and Functionals:

Systems have an input, $x(t)$, and an output, $y(t)$. The input may be periodic, transient, or random. There are two ways we can view the relation between input and output:

Operators:

An *operator* is a rule for transforming one function into another:

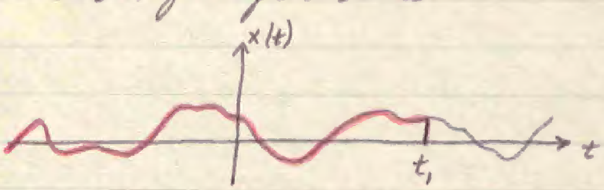
$$x(t) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y(t) \quad ; \quad y(t) = \mathcal{H}[x(t)]$$

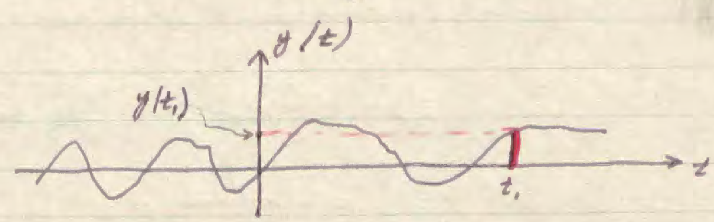
Here, $y(t)$ is the function produced from the function $x(t)$ according to the rule specified by $\mathcal{H}$.

By a "function", we mean the entire set of values taken on over all values of time. Thus, $x(t_1)$ is the value of the function $x(\cdot)$ at time t_1.

Functionals:

A functional is a number produced from a function according to a specified rule:





$y(t_1)$ is a number which is determined by & is dependent on ~~all values of~~ $x(t)$, for all $t \leq t_1$.

For a linear system, these two representations are

$$y(t) = \int_{-\infty}^{t} h(\tau)\, x(t-\tau)\, d\tau \qquad \text{, functional expression}$$

$$y(t) = \mathcal{H}[x(t)] = \left[a \frac{d^2}{dt^2} + b \frac{d}{dt} + c\right][x(t)] \qquad \text{, operational expression}$$

The difference between the two representations is chiefly one of interpretation, but the functional approach avoids certain trouble spots.

Advantages of functional representation:

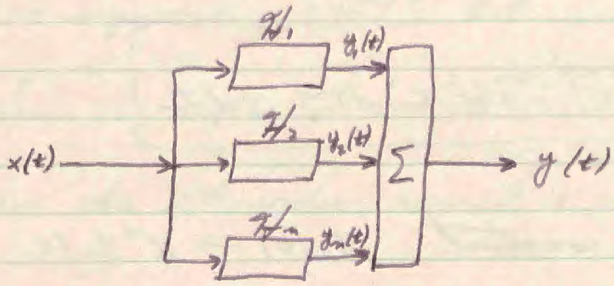
(1) The functional expressions give $y(t)$ in an explicit form; the operator (or differential equation) approach is implicit.
(2) It provides a method for dealing with combinations of systems which the operator approach cannot easily handle.
(3) It is amenable to random inputs.
(4) A generalized Fourier transform can be applied.
(5) It is convenient for physical interpretation of the results.

The operational description:

$$y(t) = y_1(t) + y_2(t) + \cdots + y_n(t) + \cdots$$

$$= \mathcal{H}_1[x(t)] + \mathcal{H}_2[x(t)] + \cdots + \mathcal{H}_n[x(t)] + \cdots = \sum_{n=1}^{\infty} \mathcal{H}_n[x(t)]$$

where $\mathcal{H}_n$ is the n^{th}-order operator



Functional formulation:

This formulation of a system is based on the mathematics in Theory of Functionals, V. Volterra, Blackie & Sons, London, 1900.

$$y(t) = y_1(t) + y_2(t) + \cdots + y_n(t) + \cdots$$

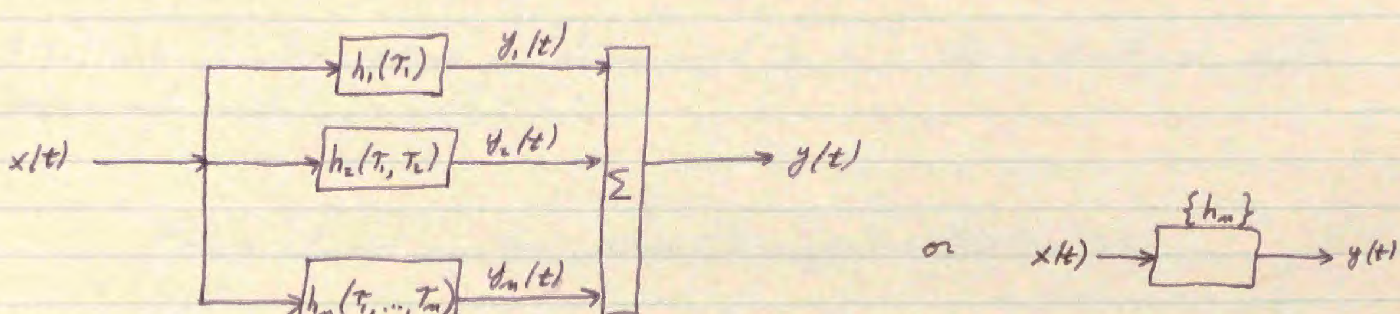
$$y(t) = \int_{-\infty}^{\infty} h_1(\tau_1)\,x(t-\tau_1)\,d\tau_1 + \iint_{-\infty}^{\infty} h_2(\tau_1,\tau_2)\,x(t-\tau_1)\,x(t-\tau_2)\,d\tau_1\,d\tau_2 + \cdots$$
$$+ \int_{-\infty}^{\infty}\!\!\cdots\!\int h_n(\tau_1,\ldots,\tau_n)\,x(t-\tau_1)\cdots x(t-\tau_n)\,d\tau_1\cdots d\tau_n + \cdots$$

The $y_n(t)$ are called functionals of the Volterra type.

The $h_n(\tau_1,\ldots,\tau_n)$ ~~are~~ is the n^{th}-order kernel, or the n^{th}-order unit impulse response.

$$\boxed{y(t) = \sum_{n=1}^{\infty} \int_{-\infty}^{\infty}\!\!\cdots\!\int_{-\infty}^{\infty} h_n(\tau_1,\ldots,\tau_n)\,x(t-\tau_1)\cdots x(t-\tau_n)\,d\tau_1,\ldots,d\tau_n}$$

Note that if $x \to Kx$, where K is a constant, $y_n(t) \to K^n y_n(t)$.



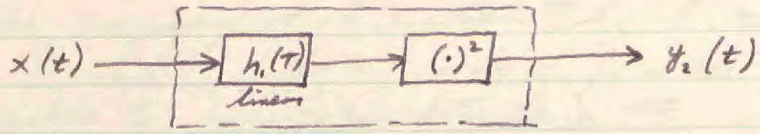
Note that for the system to be physically realizable,

$$h_n(\tau_1,\ldots,\tau_n) = 0, \quad \text{for any } \tau_i = 0, \quad i = 1,2,\ldots,n.$$

Symmetrical kernels:

The n^{th} order kernel $h_n(\tau_1, \ldots, \tau_n)$ is symmetric if the value remains unchanged for any permutation of the arguments $\tau_1, \ldots, \tau_n$. For example, $h_2(\tau_1, \tau_2)$ is symmetric if $h_2(\tau_1, \tau_2) = h_2(\tau_2, \tau_1)$.

Example:



$$y_2(t) = \left[\int h_1(\tau)x(t-\tau)d\tau\right]^2 = \iint h_1(\tau_1)h_1(\tau_2)x(t-\tau_1)x(t-\tau_2)d\tau_1 d\tau_2$$

$$= \iint h_2(\tau_1, \tau_2)\, x(t-\tau_1)x(t-\tau_2)d\tau_1 d\tau_2$$

where $\boxed{h_2(\tau_1, \tau_2) = h_1(\tau_1)h_1(\tau_2)}$

Now suppose $h_1(\tau) = E e^{-a\tau}$, $\tau \geq 0$

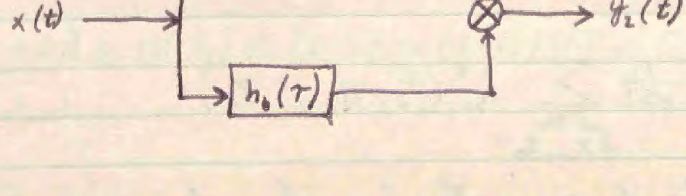
then $h_2(\tau_1, \tau_2) = E^2 e^{-a(\tau_1+\tau_2)} = h_2(\tau_2, \tau_1)$

Thus this 2^{nd}-order kernel is symmetric.

Unsymmetric kernels:

An unsymmetric kernel is a kernel that is not symmetric. An unsymmetric kernel is denoted by an asterisk, as $h_2^*(\tau_1, \tau_2)$.

Example:



$h_a(\tau) \neq h_b(\tau)$

$$y_2(t) = \iint h_a(\tau_1) h_b(\tau_2) x(t-\tau_1) x(t-\tau_2) d\tau_1 d\tau_2$$

$$\boxed{h_2^*(\tau_1, \tau_2) = h_a(\tau_1) h_b(\tau_2)}$$

Suppose $\begin{cases} h_a(\tau) = E e^{-a\tau}, & \tau \geq 0 \\ h_b(\tau) = E e^{-b\tau}, & \tau \geq 0 \end{cases}$

$$\boxed{h_2^*(\tau_1, \tau_2) = E^2 e^{-a\tau_1 - b\tau_2} \quad \begin{matrix} \tau_1 \geq 0 \\ \tau_2 \geq 0 \end{matrix}} \neq h_2(\tau_2, \tau_1) = E^2 e^{-a\tau_2 - b\tau_1}$$

"Symmetrization":

It will be later shown that it is preferrable to work with symmetric kernels. We can always reduce an unsymmetric kernel to a symmetric one by "averaging" the kernel with all possible permutations of the arguments

For example, in the above case,

$$h_2(\tau_1, \tau_2) \equiv \frac{1}{2}\left[h_2^*(\tau_1, \tau_2) + h_2^*(\tau_2, \tau_1)\right]$$

Note that

$$y_2(t) = \iint h_2^*(\tau_1, \tau_2) x(t-\tau_1) x(t-\tau_2) d\tau_1 d\tau_2 = \iint h_2^*(\tau_2, \tau_1) x(t-\tau_1) x(t-\tau_2) d\tau_1 d\tau_2$$

$$= \frac{1}{2}\iint \left[h_2^*(\tau_1, \tau_2) + h_2^*(\tau_2, \tau_1)\right] x(t-\tau_1) x(t-\tau_2) d\tau_1 d\tau_2$$

$$= \iint h(\tau_1, \tau_2) x(t-\tau_1) x(t-\tau_2) d\tau_1 d\tau_2 .$$

In general,

$$h_n(\tau_1, \dots, \tau_n) = \frac{1}{n!} \sum_{\substack{\text{all } P_n \\ \text{of } \tau_1, \dots, \tau_n}} h_n^*(\tau_i, \tau_j, \dots, \tau_k) \quad , \quad i, j, \dots, k = 1, 2, \dots, n$$

Henceforward, we will treat only symmetric kernels unless specified otherwise.

Physical interpretation of kernels:

First-order:

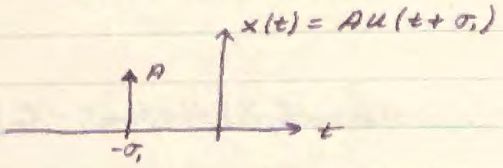
$$y_1(t) = \int h_1(\tau_1)\, x(t-\tau_1)\, d\tau_1$$

From our past experience of linear systems, we know that the first order kernel $h_1(\tau)$ can be interpreted as the ~~[illegible]~~ unit impulse response of the system. So if $x(t) = u_1(t)$, then $y_1(t) = h_1(t)$.

A slightly different way of formulating the problem is extendable to higher order kernels:

$x(t) \longrightarrow \boxed{h_1(\tau_1)} \longrightarrow y_1(t)$ Let $x(t) = A\,u(t+\sigma_1)$

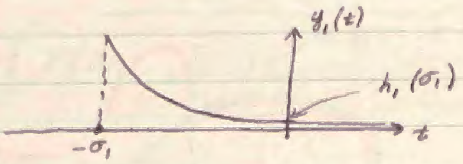
(where $u(t)$ is the unit impulse at $t=0$).



$$y_1(t) = \int h_1(\tau_1)\, x(t-\tau_1)\, d\tau_1 = \int h_1(\tau_1)\, A\,u(t+\sigma_1-\tau_1)\, d\tau_1 = A\, h_1(t+\sigma_1)$$

In particular, if $A = 1$, and we look at the output at time $t=0$,

$$\underline{y_1(0) = h_1(\sigma_1)}$$



Second-order:

$$y_2(t) = \iint h_2(\tau_1,\tau_2)\, x(t-\tau_1)\, x(t-\tau_2)\, d\tau_1\, d\tau_2$$

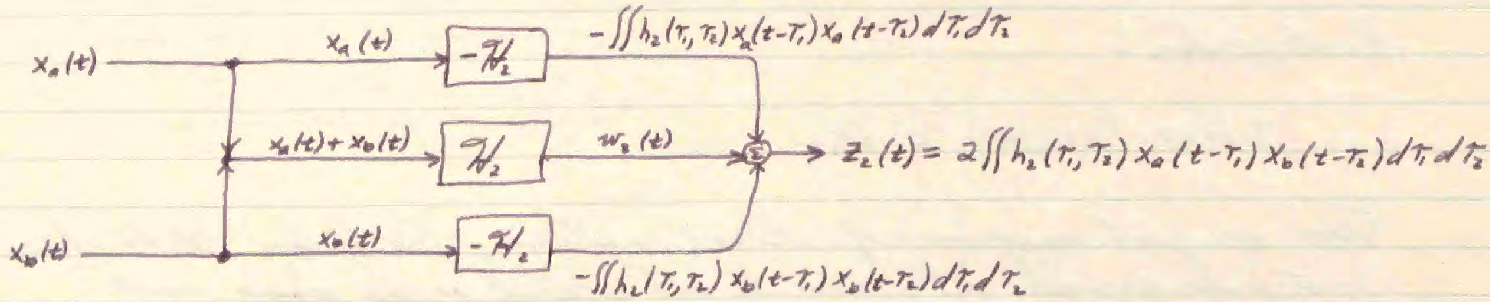
Now suppose $x(t) = x_a(t) + x_b(t) \longrightarrow \boxed{H_2} \longrightarrow w_2(t)$:

$$w_2(t) = \iint h_2(\tau_1,\tau_2)\, x_a(t-\tau_1)\, x_a(t-\tau_2)\, d\tau_1\, d\tau_2$$

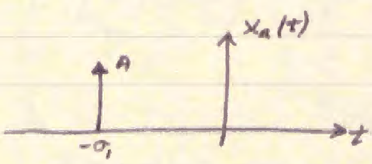
$$+ \iint h_2(\tau_1,\tau_2)\, x_b(t-\tau_1)\, x_b(t-\tau_2)\, d\tau_1\, d\tau_2$$

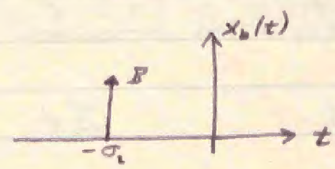
$$+ 2\iint h_2(\tau_1,\tau_2)\, x_a(t-\tau_1)\, x_b(t-\tau_2)\, d\tau_1\, d\tau_2$$

We can now build the following system:



Let $\quad x_a(t) = A\,u(t+\sigma_1) \quad , \quad x_b(t) = B\,u(t+\sigma_2)$

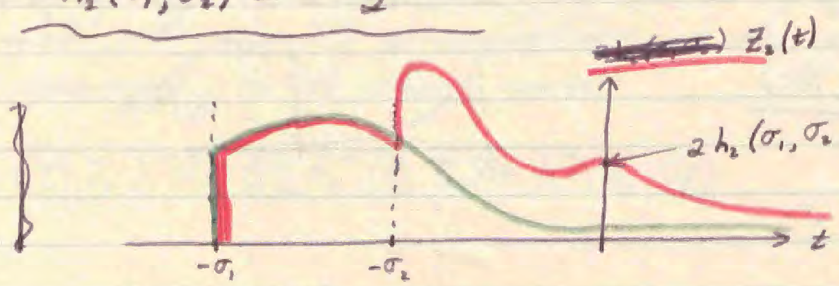




$$z_2(t) = 2\iint h_2(\tau_1,\tau_2)\,A\,u(t+\sigma_1-\tau_1)\,B\,u(t+\sigma_2-\tau_2)\,d\tau_1 d\tau_2 = \underline{2AB\,h_2(t+\sigma_1,\,t+\sigma_2)}$$

If we now let $A=B=1$, and observe the output at $t=0$,

$$\underline{h_2(\sigma_1,\sigma_2) = \frac{z_2(0)}{2}}$$



System response to constant input:

$x(t) \longrightarrow \boxed{\{h_n\}} \longrightarrow y(t)$ Let $x(t) = E = \text{const}$

$$y(t) = \sum_{n=1}^{\infty} \int \cdots \int h_n(\tau_1, \ldots, \tau_n)\, x(t-\tau_1) \cdots x(t-\tau_n)\, d\tau_1 \cdots d\tau_n$$

$$= \sum_{n=1}^{\infty} \int \cdots \int h_n(\tau_1, \ldots, \tau_n)\, E^n\, d\tau_1 \cdots d\tau_n$$

$$\boxed{y(t) = \sum_{n=1}^{\infty} a_n E^n}$$

where $a_n = \int \cdots \int h_n(\tau_1, \ldots, \tau_n)\, d\tau_1 \cdots d\tau_n$

System response to transient input:

Here, we introduce a multi-dimensional Fourier transform, defined as follows:

$$F(\omega_1, \ldots, \omega_n) \equiv \frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(t_1, \ldots, t_n)\, e^{-j\omega_1 t_1 - \cdots - j\omega_n t_n}\, dt_1 \cdots dt_n$$

$$f(t_1, \ldots, t_n) \equiv \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} F(\omega_1, \ldots, \omega_n)\, e^{j\omega_1 t_1 + \cdots + j\omega_n t_n}\, d\omega_1 \cdots d\omega_n$$

Now define $y_{(2)}(t_1, t_2) = \iint h_2(\tau_1, \tau_2)\, x(t_1 - \tau_1)\, x(t_2 - \tau_2)\, d\tau_1\, d\tau_2$

This will be used as an artifice to get $y_2(t)$ in a form we can use the multi-dimensional transform on. Note $y_{(2)}(t,t) = y_2(t)$.

$$Y_{(2)}(\omega_1, \omega_2) = \frac{1}{(2\pi)^2} \iint e^{-j\omega_1 t_1 - j\omega_2 t_2}\, dt_1\, dt_2 \iint h_2(\tau_1, \tau_2)\, x(t_1 - \tau_1)\, x(t_2 - \tau_2)\, d\tau_1\, d\tau_2$$

Now let $t_1 - \tau_1 = u$, $t_2 - \tau_2 = v$

$$Y_{(2)}(\omega_1, \omega_2) = \frac{1}{(2\pi)^2} \iint e^{-j\omega_1 (u+\tau_1) - j\omega_2 (v+\tau_2)}\, du\, dv \iint h_2(\tau_1, \tau_2)\, x(u)\, x(v)\, d\tau_1\, d\tau_2$$

$$Y_2(\omega_1, \omega_2) = \iint h_2(\tau_1, \tau_2) e^{-j\omega_1\tau_1 - j\omega_2\tau_2} d\tau_1 d\tau_2 \quad \frac{1}{2\pi}\int x(u) e^{-j\omega_1 u} du \quad \frac{1}{2\pi}\int x(v) e^{-j\omega_2 v} dv$$

Hence,

$$Y_{(2)}(\omega_1, \omega_2) = H_2(\omega_1, \omega_2) X(\omega_1) X(\omega_2)$$

where $$H_2(\omega_1, \omega_2) = \iint h_2(\tau_1, \tau_2) e^{-j\omega_1\tau_1 - j\omega_2\tau_2} d\tau_1 d\tau_2$$

$$X(\omega) = \frac{1}{2\pi}\int x(t) e^{-j\omega t} dt.$$

The system transfer function has no $\left(\frac{1}{2\pi}\right)$ factor as it is interpreted as the impulse response of the system. This enters in because Lee has put his $\left(\frac{1}{2\pi}\right)$ with the transform $F(\omega) = \frac{1}{2\pi}\int f(t) e^{-j\omega t} dt$, as opposed to the usual practice of putting $F(\omega) = \int f(t) e^{-j\omega t} dt$.

<u>Finding $Y_2(\omega)$</u>:

$$y_2(t) = \int Y_2(\omega) e^{j\omega t} d\omega$$

$$= y_{(2)}(t,t) = \iint Y_{(2)}(\omega_1, \omega_2) e^{j(\omega_1+\omega_2)t} d\omega_1 d\omega_2$$

$$= \int e^{j\omega t} d\omega \int Y_{(2)}(\omega_1, \omega - \omega_1) d\omega_1 \qquad \text{if} \quad \omega_2 = \omega - \omega_1$$

Thus, by identifying terms in these last two expressions, we get

$$\boxed{Y_2(\omega) = \int Y_{(2)}(\omega_1, \omega - \omega_1) d\omega_1}$$

In practice then, we would find $h_2(\tau_1, \tau_2)$; transform it ~~of~~ to ~~get $Y_2(\omega_1, \omega_2)$; find $Y_2(\omega)$; and inverse~~ get $H_2(\omega_1, \omega_2)$; multiply by $X(\omega_1)X(\omega_2)$ to get $Y_{(2)}(\omega_1, \omega_2)$; find $Y_2(\omega)$; and inverse transform this ~~fn~~ to find $y_2(t)$.

Example: $x(t) \longrightarrow \boxed{h_2(\tau_1,\tau_2)} \longrightarrow y_2(t)$

$$x(t) = A\,u(t)$$
$$h_2(\tau_1,\tau_2) = E^2 e^{-a(\tau_1+\tau_2)} \quad ; \tau_1,\tau_2 \geq 0.$$

$$H_2(\omega_1,\omega_2) = \int_0^\infty\!\!\int_0^\infty E^2 e^{-a(\tau_1+\tau_2)} e^{-j\omega_1\tau_1 - j\omega_2\tau_2}\, d\tau_1\, d\tau_2$$

$$= E^2 \int_0^\infty e^{-a\tau_1 - j\omega_1\tau_1} d\tau_1 \int_0^\infty e^{-a\tau_2 - j\omega_2\tau_2} d\tau_2 \;=\; E^2 \frac{1}{(a+j\omega_1)(a+j\omega_2)}$$

$$X(\omega) = \frac{A}{2\pi}$$

so $\quad Y_{(2)}(\omega_1,\omega_2) = H_2(\omega_1,\omega_2)\,X(\omega_1)X(\omega_2) = \dfrac{A^2E^2}{(2\pi)^2}\,\dfrac{1}{(a+j\omega_1)(a+j\omega_2)}$

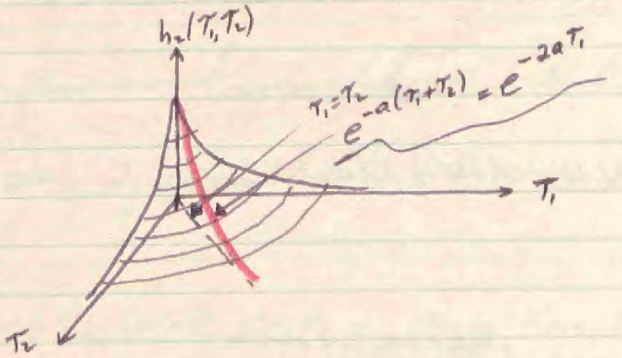
$$Y_2(\omega) = \int Y_{(2)}(\omega_1,\omega-\omega_1)\,d\omega_1 \;=\; \frac{A^2E^2}{(2\pi)^2}\int \frac{d\omega_1}{[a+j\omega_1][a+j(\omega-\omega_1)]}$$

Now, we know that convolution in the frequency domain is equivalent to multiplication in the time domain:

$$F(\omega) = \frac{1}{a+j\omega} \;\leftrightarrow\; f(t) = \int F(\omega)e^{j\omega t}d\omega = \int_{-\infty}^{\infty} \frac{e^{j\omega t}}{a+j\omega}\,d\omega \;=\; e^{-at}\int_{-\infty}^{\infty}\frac{e^{(a+j\omega)t}}{a+j\omega}\,d\omega$$

$$f(t) = e^{-at}\,2\pi \sum \mathrm{Res}[\;] = 2\pi e^{-at} \quad , t \geq 0$$

So
$$\boxed{Y_2(\omega) = \frac{A^2E^2}{2\pi^2}\left[2\pi e^{-at}\right]^2 = A^2E^2 e^{-2at} \quad , t \geq 0}$$



Review of convolution in frequency domain:

If $\underline{G(\omega) = \int F_a(u) F_b(\omega - u)\, du}$

then $g(t) = \int G(\omega) e^{j\omega t} d\omega = \int e^{j\omega t} d\omega \int F_a(u) F_b(\omega - u)\, du$

Letting $\omega - u = v$,

$$g(t) = \int e^{j(u+v)t} dv \int F_a(u) F_b(v)\, du = \int F_a(u) e^{jut} du \int F_b(v) e^{jvt} dv$$

$$\underline{g(t) = f_a(t) f_b(t)}$$

In our case on the last page, $f_a(t) = f_b(t)$.

Finding the general transform of the n^{th} order output $Y_n(\omega)$:

First, we will find $Y_3(\omega)$ & then extrapolate to $Y_n(\omega)$:

$$y_3(t) = \iiint h_3(\tau_1, \tau_2, \tau_3)\, x(t-\tau_1)\, x(t-\tau_2)\, x(t-\tau_3)\, d\tau_1 d\tau_2 d\tau_3$$

As before, let us define

$$y_{(3)}(t_1, t_2, t_3) \equiv \iiint h_3(\tau_1, \tau_2, \tau_3)\, x(t_1-\tau_1)\, x(t_2-\tau_2)\, x(t_3-\tau_3)\, d\tau_1 d\tau_2 d\tau_3$$

Transforming gives

$$Y_{(3)}(\omega_1, \omega_2, \omega_3) = H_3(\omega_1, \omega_2, \omega_3)\, X(\omega_1)\, X(\omega_2)\, X(\omega_3).$$

Now

$$y_{(3)}(t_1, t_2, t_3) = \iiint Y_{(3)}(\omega_1, \omega_2, \omega_3)\, e^{j\omega_1 t_1 + j\omega_2 t_2 + j\omega_3 t_3}\, d\omega_1 d\omega_2 d\omega_3$$

and $y_3(t) = y_{(3)}(t, t, t)$

First let $\begin{cases} t_1 = t_2 = t \\ \omega_1 + \omega_2 = \omega_4 \end{cases}$

$$y_{(3)}(t, t, t_3) = \iiint Y_{(3)}(\omega_1, \omega_4 - \omega_1, \omega_3)\, e^{j\omega_4 t + j\omega_3 t_3}\, d\omega_1\, d\omega_4\, d\omega_3$$

Now let $\begin{cases} t_3 = t_1 \\ \omega_4 + \omega_3 = \omega \end{cases}$

$$y_{(3)}(t, t, t) = \iiint Y_{(3)}(\omega_1, \omega_4 - \omega_1, \omega - \omega_4)\, e^{j\omega t}\, d\omega_1\, d\omega_4\, d\omega$$

$$= \int e^{j\omega t} d\omega \underline{\iint Y_{(3)}(\omega_1, \omega_4 - \omega_1, \omega - \omega_4)\, d\omega_1\, d\omega_4}$$

$$= y_3(t) = \int e^{j\omega t}\, Y_3(\omega)\, d\omega$$

$$\Rightarrow \quad Y_3(\omega) = \iint Y_{(3)}(\omega_1, \omega_4 - \omega_1, \omega - \omega_4)\, d\omega_1\, d\omega_4$$

But now notice that since $h_3(\tau_1, \tau_2, \tau_3)$ is symmetric, $H_3(\omega_1, \omega_2, \omega_3)$ must be symmetric. Also $X(\omega_1)X(\omega_2)X(\omega_3)$ is symmetric, so $Y_{(3)}(\omega_1, \omega_2, \omega_3)$ must be symmetric. Hence, any re-arrangement of variables is immaterial & letting $\omega_4 \to u_1$, $\omega_1 \to u_2$, we have

$$\boxed{Y_3(\omega) = \iint Y_{(3)}(\omega - u_1, u_1 - u_2, u_2)\, du_1\, du_2}$$

This is a two-dimensional "convolution" of sorts.

In general, then, by extrapolating from our results for $Y_2(\omega)$ + $Y_3(\omega)$,

$$\boxed{Y_n(\omega) = \int \cdots \int Y_{(n)}(\omega - u_1, u_1 - u_2, \ldots, u_{n-2} - u_{n-1}, u_{n-1})\, du_1 \cdots du_{n-1}}$$

where

$$Y_{(n)}(\omega_1, \ldots, \omega_n) = H_n(\omega_1, \ldots, \omega_n)\, X(\omega_1) \cdots X(\omega_n)$$

Response to sinusoidal input:

First-order: $y_1(t) = \int h_1(\tau_1)x(t-\tau_1)d\tau_1$

$$x(t) = E_m \cos(\omega t+\theta) = \mathrm{Re}\, E_m e^{j\theta} e^{j\omega t} = \mathrm{Re}\, E e^{j\omega t}, \quad E = E_m e^{j\theta}$$

$$y_1(t) = \int h_1(\tau_1)\,\mathrm{Re}\, E e^{j\omega(t-\tau_1)} d\tau_1 = \mathrm{Re}\, E e^{j\omega t}\int h_1(\tau_1) e^{-j\omega\tau_1} d\tau_1$$

$$\boxed{y_1(t) = \mathrm{Re}\; E H_1(\omega) e^{j\omega t}}$$

Second-order: $y_2(t) = \iint h_2(\tau_1,\tau_2)x(t-\tau_1)x(t-\tau_2)d\tau_1 d\tau_2$

$$x(t) = \mathrm{Re}\, E e^{j\omega t} = \tfrac{1}{2}E e^{j\omega t} + \tfrac{1}{2}\bar{E} e^{-j\omega t} = x_a(t) + x_b(t)$$

$$y_2(t) = \mathcal{H}_2[x_a(t)] + \mathcal{H}_2[x_b(t)] + 2\mathcal{H}_2[x_a(t), x_b(t)]$$

$$\mathcal{H}_2[x_a(t)] = \iint h_2(\tau_1,\tau_2)\frac{E^2}{4} e^{j\omega(t-\tau_1)} e^{j\omega(t-\tau_2)} d\tau_1 d\tau_2$$

$$= \frac{E^2}{4} e^{j2\omega t}\iint h_2(\tau_1,\tau_2) e^{-j\omega\tau_1 - j\omega\tau_2} d\tau_1 d\tau_2$$

$$\boxed{\mathcal{H}_2[x_a(t)] = \frac{E^2}{4} H_2(\omega,\omega) e^{j2\omega t}}$$

$$\mathcal{H}_2[x_b(t)] = \frac{\bar{E}^2}{4}\iint h_2(\tau_1,\tau_2) e^{-j\omega(t-\tau_1)} e^{-j\omega(t-\tau_2)} d\tau_1 d\tau_2$$

$$\boxed{\mathcal{H}_2[x_b(t)] = \frac{\bar{E}^2}{4} H_2(-\omega,-\omega) e^{-j2\omega t} = \overline{\mathcal{H}_2[x_a(t)]}}$$

Note $\underline{H_2(-\omega_1,-\omega_2) = \iint h_2(\tau_1,\tau_2) e^{+j\omega_1\tau_1 + j\omega_2\tau_2} d\tau_1 d\tau_2 = \overline{H_2(\omega_1,\omega_2)}}$ since $h_2(\tau_1,\tau_2)$ is real

$$\mathcal{H}_2[x_a(t), x_b(t)] = \frac{E\bar{E}}{4}\iint h_2(\tau_1,\tau_2) e^{j\omega(t-\tau_1) - j\omega(t-\tau_2)} d\tau_1 d\tau_2$$

$$= \frac{|E|^2}{4}\iint h_2(\tau_1,\tau_2) e^{-j\omega\tau_1 + j\omega\tau_2} d\tau_1 d\tau_2$$

$$\boxed{\mathcal{H}_2[x_a(t), x_b(t)] = \frac{|E|^2}{4} H_2(\omega,-\omega)}$$

Now, adding these terms, we get

$$y_2(t) = \frac{E^2}{4} H_2(\omega,\omega) e^{j2\omega t} + \frac{\bar{E}^2}{4} H_2(-\omega,-\omega) e^{-j2\omega t} + \frac{|E|^2}{2} H_2(\omega,-\omega)$$

$$\boxed{y_2(t) = \mathrm{Re}\, \frac{E^2}{2} H_2(\omega,\omega) e^{j2\omega t} + \frac{|E|^2}{2} H_2(\omega,-\omega)}$$

In general, $H_2(\omega,\omega)$ is complex. But due to the symmetry of the kernel $h_2(\tau_1,\tau_2)$, $H_2(\omega,-\omega)$ is real:

$$H_2(\omega,-\omega) = \iint h_2(\tau_1,\tau_2) e^{-j\omega(\tau_1-\tau_2)} d\tau_1 d\tau_2 = \iint h_2(\tau_1,\tau_2) e^{-j\omega(\tau_2-\tau_1)} d\tau_1 d\tau_2$$

$$H_2(\omega,-\omega) = H(-\omega,\omega) = \bar{H}(\omega,-\omega)$$

The only way $H_2 = \bar{H}_2$ is for H_2 to be real.

<u>Example</u>: $x(t) \longrightarrow \boxed{h_1(\tau_1)} \longrightarrow \boxed{(\cdot)^2} \longrightarrow y_2(t)$

$$x(t) = \mathrm{Re}\, E e^{j\omega t}, \quad h_1(t) = A e^{-at},\ t \geq 0$$

$$h_2(\tau_1,\tau_2) = A^2 e^{-a(\tau_1+\tau_2)} \quad ; \ \tau_1,\tau_2 \geq 0$$

$$H_2(\omega,\omega) = A^2 \iint_0^\infty e^{-a(\tau_1+\tau_2)} e^{-j\omega\tau_1 - j\omega\tau_2} d\tau_1 d\tau_2 = A^2 \int_0^\infty e^{-a\tau_1} e^{-j\omega\tau_1} d\tau_1 \int_0^\infty e^{-a\tau_2} e^{-j\omega\tau_2} d\tau_2$$

$$\underline{H_2(\omega,\omega) = A^2 \frac{1}{(a+j\omega)^2}}$$

$$H_2(\omega,-\omega) = A^2 \left(\frac{1}{a+j\omega}\right)\left(\frac{1}{a-j\omega}\right) = \frac{A^2}{a^2+\omega^2}$$

So, $$\boxed{y_2(t) = \mathrm{Re}\left\{\frac{E^2}{2} e^{j2\omega t} \frac{A^2}{(a+j\omega)^2}\right\} + \frac{|E|^2 A^2}{2(a^2+\omega^2)}}$$

<u>Third-order</u>: $y_3(t) = \iiint h_3(\tau_1,\tau_2,\tau_3)\, x(t-\tau_1)\, x(t-\tau_2)\, x(t-\tau_3)\, d\tau_1\, d\tau_2\, d\tau_3$

Again, $x(t) = \frac{E}{2}e^{j\omega t} + \frac{\bar{E}}{2}e^{-j\omega t} = x_a(t) + x_b(t)$

$$\underline{y_3(t) = \mathcal{H}_3[x_a(t)] + \mathcal{H}_3[x_b(t)] + 3\mathcal{H}_3[x_a(t), x_a(t), x_b(t)] + 3\mathcal{H}_3[x_a(t), x_b(t), x_b(t)]}$$

$$\mathcal{H}_3[x_a(t)] = \iiint h_3(\tau_1,\tau_2,\tau_3)\, \frac{E^3}{2^3}\, e^{j\omega(t-\tau_1)+j\omega(t-\tau_2)+j\omega(t-\tau_3)}\, d\tau_1\, d\tau_2\, d\tau_3$$

$$= \frac{E^3}{2^3} e^{j3\omega t} \iiint h_3(\tau_1,\tau_2,\tau_3)\, e^{-j\omega\tau_1 - j\omega\tau_2 - j\omega\tau_3}\, d\tau_1\, d\tau_2\, d\tau_3$$

$$\boxed{\mathcal{H}_3[x_a(t)] = \frac{E^3}{2^3} H_3(\omega,\omega,\omega)\, e^{j3\omega t}}$$

Similarly,

$$\boxed{\mathcal{H}_3[x_b(t)] = \frac{\bar{E}^3}{2^3} H_3(-\omega,-\omega,-\omega)\, e^{-j3\omega t}}$$

$$\boxed{\mathcal{H}_3[x_a(t), x_a(t), x_b(t)] = \frac{E^2\bar{E}}{2^3} H_3(\omega,\omega,-\omega)\, e^{j\omega t}}$$

$$\boxed{\mathcal{H}_3[x_a(t), x_b(t), x_b(t)] = \frac{E\bar{E}^2}{2^3} H_3(\omega,-\omega,-\omega)\, e^{-j\omega t}}$$

So

$$\boxed{y_3(t) = \mathrm{Re}\left\{ \frac{E^3}{2^2} H_3(\omega,\omega,\omega)\, e^{j3\omega t} + 3\,\frac{|E|^2 E}{2^2} H_3(\omega,\omega,-\omega)\, e^{j\omega t} \right\}}$$

Obviously, a kernel of order n will give harmonics of the fundamental frequency of order $n, n-2, n-4, \ldots$ Thus an odd kernel gives only odd harmonics, and an even kernel gives only even harmonics

A result of this is that if this representation of non-linear systems is correct, sub-harmonics are <u>not</u> generated by non-linearities in a time-invariant system.

System response to random inputs:

Mean value of output:

First order: $y_1(t) = \int h_1(\tau_1)\, x(t-\tau_1)\, d\tau_1$

$$\overline{y_1(t)} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} dt \int h_1(\tau_1)\, x(t-\tau_1)\, d\tau_1 = \int h_1(\tau_1)\, d\tau_1 \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t-\tau_1)\, dt$$

$$\boxed{\overline{y_1(t)} = \int h_1(\tau_1)\, \overline{x(t-\tau_1)}\, d\tau_1 = \overline{x(t)} \int h_1(\tau_1)\, d\tau_1}$$

Second order:

$$\overline{y_2(t)} = \iint h_2(\tau_1,\tau_2)\, \overline{x(t-\tau_1)\, x(t-\tau_2)}\, d\tau_1\, d\tau_2$$

$$\boxed{\overline{y_2(t)} = \iint h_2(\tau_1,\tau_2)\, \varphi_{xx}(\tau_1-\tau_2)\, d\tau_1\, d\tau_2}$$

depends on autocorrelation function of the input & therefore only on the second moments of the probability distribution.

Third order:

$$\overline{y_3(t)} = \iiint h_3(\tau_1,\tau_2,\tau_3)\, \overline{x(t-\tau_1)\, x(t-\tau_2)\, x(t-\tau_3)}\, d\tau_1\, d\tau_2\, d\tau_3$$

$$\boxed{\overline{y_3(t)} = \iiint h_3(\tau_1,\tau_2,\tau_3)\, \varphi_{xxx}(\tau_1-\tau_2,\ \tau_2-\tau_3)\, d\tau_1\, d\tau_2\, d\tau_3}$$

where

$$\boxed{\varphi_{xxx}(\tau_1,\tau_2) \equiv \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)\, x(t+\tau_1)\, x(t+\tau_1+\tau_2)\, dt}$$

Mean n^{th}-order output:

By ~~extrapolo~~ extrapolation of the observed trend:

$$\boxed{\overline{y_n(t)} = \int\cdots\int \varphi_{nx}(\tau_1-\tau_2,\ \tau_2-\tau_3,\ \ldots,\ \tau_{n-1}-\tau_n)\, d\tau_1 \cdots d\tau_n}$$

$$\boxed{\overline{y(t)} = \sum_{n=1}^{\infty} \overline{y_n(t)}}$$

[Second & higher autocorrelation functions are discussed in "Properties of the Second Order Autocorrelation Functions" by H. Hayase, Technical Report (RLE) #330, 1957, and in "Measurement of Correlation Functions by M. Schetzen, RLE Quarterly Progress Report, October 1960.

Note that the third-order correlation function could be defined as $\overline{x(t)x(t+\tau_1)x(t+\tau_2)}$. $(\tau_1 \to \tau_1), (\tau_1+\tau_2 \to \tau_2)$.]

Autocorrelation of the output:

$$\boxed{\overline{y(t)y(t+T)} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \overline{y_m(t)y_n(t+T)}}$$

For example,

$$\overline{y_1(t)y_1(t+T)} = \iint h_1(\tau_1)h_1(\sigma_1)\,\overline{x(t-\tau_1)x(t+T-\sigma_1)}\,d\tau_1 d\sigma_1$$

$$= \iint h_1(\tau_1)h_1(\sigma_1)\,\varphi_{xx}(\tau_1+T-\sigma_1)\,d\tau_1 d\sigma_1$$

$$\boxed{\overline{y_1(t)y_1(t+T)} = \iint h_1(\tau_1)h_1(T+\tau_2)\,\varphi_{xx}(\tau_1-\tau_2)\,d\tau_1 d\tau_2}\quad , \; T-\sigma_1 = -\tau_2$$

$$\overline{y_2(t)y_2(t+T)} = \iiiint h_2(\tau_1,\tau_2)h_2(\sigma_1,\sigma_2)\,\overline{x(t-\tau_1)x(t-\tau_2)x(t+T-\sigma_1)x(t+T-\sigma_2)}\,d\tau_1 d\tau_2 d\sigma_1 d\sigma_2$$

$$= \iiiint h_2(\tau_1,\tau_2)h_2(\tau_3+T,\tau_4+T)\,\overline{x(t-\tau_1)x(t-\tau_2)x(t-\tau_3)x(t-\tau_4)}\,d\tau_1 d\tau_2 d\tau_3 d\tau_4$$

by $\tau_3 = \sigma_1 - T$, $\tau_4 = \sigma_2 - T$

so $$\boxed{\overline{y_2(t)y_2(t+T)} = \iiiint h_2(\tau_1,\tau_2)h_2(\tau_3+T,\tau_4+T)\,\varphi_{4x}(\tau_1-\tau_2,\tau_2-\tau_3,\tau_3-\tau_4)\,d\tau_1 d\tau_2 d\tau_3 d\tau_4}$$

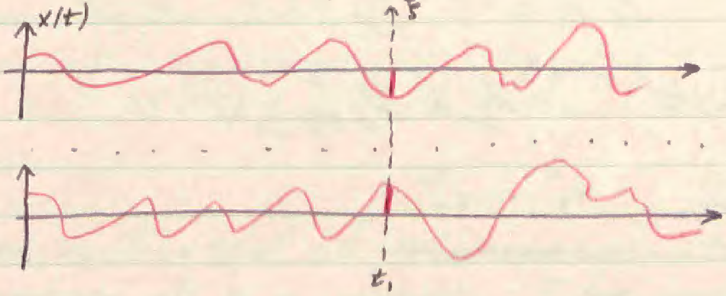
Obviously, by extension

$$\boxed{\overline{y_n(t)y_n(t+T)} = \int\cdots\int h_n(\tau_1,\ldots,\tau_n)h_n(\tau_{n+1}+T,\ldots,\tau_{2n}+T)\,\varphi_{2nx}(\tau_1-\tau_2,\ldots,\tau_{2n-1}-\tau_{2n})\,d\tau_1\cdots d\tau_{2n}}$$

and $\rightarrow$ $$\boxed{\overline{y_m(t)y_n(t+T)} = \int\cdots\int h_m(\tau_1,\ldots,\tau_m)h_n(\tau_{m+1}+T,\ldots,\tau_{m+n}+T)\,\varphi_{(m+n)x}(\tau_1-\tau_2,\ldots,\tau_{m+n-1}-\tau_{m+n})\,d\tau_1\ldots d\tau_{m+n}}$$

Some notes on gaussian random variables:

In the Volterra formulation for a non-linear system, our averages and correlation functions will involve averages of the form $\overline{x(t-T_1)\cdots x(t-T_n)}$. For most random process in particular and certainly for random processes in general, these expressions will rapidly get very complex.
Thus we restrict our attention for the time being on gaussian random inputs $x(t)$.



$x(\cdot)$-ensemble

t_1: random variable ξ over ensemble.

Letting z be the range variable for ξ, $\quad P_\xi(z) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(z-\bar{\xi})^2}{2\sigma^2}}$

Let η be the normalized random variable $\quad \eta \equiv \frac{\xi - \bar{\xi}}{\sigma}$.

We want to consider expressions of the form $\overline{\eta_1 \eta_2 \cdots \eta_n}$:
As will be proved on the next page or so, for gaussian random variables with zero mean,

$$\boxed{\overline{\eta_1 \cdots \eta_n} = \sum_A \prod_{\text{all } i \neq j} \overline{\eta_i \eta_j} \qquad \text{if } n \text{ is even}}$$

where the sum is over A = all distinct ways of partitioning $\eta_1, \ldots, \eta_n$ into products of pairs, $\eta_i \eta_j$, $i \neq j$.

For example, $n = 2m = 4$

$$\overline{\eta_1\eta_2\eta_3\eta_4} = \overline{\eta_1\eta_2}\;\overline{\eta_3\eta_4} + \overline{\eta_1\eta_3}\;\overline{\eta_2\eta_4} + \overline{\eta_1\eta_4}\;\overline{\eta_2\eta_3}$$

The number of distinct ways of partitioning is

$$N = \frac{(2m)!}{m!\,2^m} = (2m-1)(2m-3)\cdots 1$$

The Average of the Product of Gaussian Variables:

~~[illegible]~~ This is based on a paper by the above name by M. Schetzen, RLE Quarterly Progress Report, Jan 1961.

Suppose we have the joint probability density of n gaussian random variables $\eta_1, \dots, \eta_n$, and the corresponding characteristic function

$$P_\eta(y_1, \dots, y_n) \leftrightarrow M_\eta(\alpha_1, \dots, \alpha_n) = \overline{e^{j(\alpha_1\eta_1 + \cdots \alpha_n \eta_n)}} \quad \text{, ensemble avg.}$$

$$M_\eta(\alpha_1, \dots, \alpha_n) = \int \cdots \int P_\eta(y_1, \dots, y_n) e^{j\sum_{i=1}^{n} \alpha_i y_i} dy_1 \cdots dy_n$$

$$= \sum_{K_1=0}^{\infty} \cdots \sum_{K_n=0}^{\infty} C_{K_1 \cdots K_n} \alpha_1^{K_1} \cdots \alpha_n^{K_n}$$

where

$$C_{K_1 \cdots K_n} = \frac{1}{K_1! \cdots K_n!} \frac{\partial^{K_1}}{\partial \alpha_1^{K_1}} \cdots \frac{\partial^{K_n}}{\partial \alpha_n^{K_n}} M_\eta(\alpha_1, \dots, \alpha_n) \Big|_{\alpha_1 = \cdots = \alpha_n = 0}$$

Now then,

$$\frac{\partial^{K_1}}{\partial \alpha_1^{K_1}} \cdots \frac{\partial^{K_n}}{\partial \alpha_n^{K_n}} M_\eta(\alpha_1, \dots, \alpha_n) \Big|_{\alpha_1 = \cdots = \alpha_n = 0} = j^{K_1} \cdots j^{K_n} \int \cdots \int y_1^{K_1} \cdots y_n^{K_n} P_\eta(y_1, \dots, y_n) e^{j\sum_{i=1}^{n} \alpha_i y_i} dy_1 \cdots dy_n \Big|_{\alpha_1 = \cdots = \alpha_n = 0}$$

$$= j^{(K_1 + \cdots + K_n)} \overline{\eta_1^{K_1} \cdots \eta_n^{K_n}}$$

Then we can write

$$M_\eta(\alpha_1, \dots, \alpha_n) = \sum_{K_1=0}^{\infty} \cdots \sum_{K_n=0}^{\infty} \frac{(j\alpha_1)^{K_1} \cdots (j\alpha_n)^{K_n}}{K_1! \cdots K_n!} \overline{\eta_1^{K_1} \cdots \eta_n^{K_n}}$$

We are interested in the term $\overline{\eta_1 \cdots \eta_n}$ corresponding to $K_1 = \cdots = K_n = 1$. This term of the expansion is

$$1^{st} \text{ order term} = (j\alpha_1) \cdots (j\alpha_n) \overline{\eta_1 \cdots \eta_n}$$

We will now compare this expansion with that developed by Cramér on page 310 [ca.] of his "Methods of Statistics.

$$M_\eta(\alpha_1,\dots,\alpha_n) = e^{-\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\overline{\eta_i\eta_j}\,\alpha_i\alpha_j}$$

A Taylor series expansion of e^x gives $e^x = \sum_{p=0}^{\infty}\frac{x^p}{p!}$

Thus

$$M_\eta(\alpha_1,\dots,\alpha_n) = \sum_{p=0}^{\infty}\left(-\tfrac{1}{2}\right)^p\left[\sum_{i=1}^{n}\sum_{j=1}^{n}\overline{\eta_i\eta_j}\,\alpha_i\alpha_j\right]^p$$

$$= 1+\left(-\tfrac{1}{2}\right)\sum_{k_1=1}^{n}\sum_{k_2=1}^{n}\overline{\eta_{k_1}\eta_{k_2}}\,\alpha_{k_1}\alpha_{k_2} + \frac{1}{2!}\left(-\tfrac{1}{2}\right)^2\sum_{k_1=1}^{n}\sum_{k_2=1}^{n}\sum_{k_3=1}^{n}\sum_{k_4=1}^{n}\overline{\eta_{k_1}\eta_{k_2}}\;\overline{\eta_{k_3}\eta_{k_4}}\,\alpha_{k_1}\alpha_{k_2}\alpha_{k_3}\alpha_{k_4}$$

Now, notice that the ~~α's occur~~ only ~~in~~ products of even numbers of α's occur. Thus we conclude that

$$\overline{\eta_1\cdots\eta_n} = 0 \quad \text{if } n \text{ is odd}$$

Now consider the case where $n = 2m$, $m = 1, 2, \dots$:

The terms in this expansion corresponding to the $j^{2m}\,\overline{\eta_1\cdots\eta_{2m}}\,\alpha_1\cdots\alpha_{2m}$ term in the previous expansion are those ~~[illegible]~~ of the m^{th} term:

$$\frac{1}{m!}\left(-\tfrac{1}{2}\right)^m\left[\sum_{i=1}^{2m}\sum_{j=1}^{2m}\overline{\eta_i\eta_j}\,\alpha_i\alpha_j\right]^m$$

$$= \frac{(-1)^m}{m!\,2^m}\sum_{k_1=1}^{2m}\cdots\sum_{k_{2m}=1}^{2m}\overline{\eta_{k_1}\eta_{k_2}}\cdots\overline{\eta_{k_{2m-1}}\eta_{k_{2m}}}\;\alpha_1\cdots\alpha_{2m}$$

There is some redundancy in these terms: there are

(2^m) terms like $\overline{\eta_i\eta_j} = \overline{\eta_j\eta_i}$

and $(m!)$ terms like $\overline{\eta_1\eta_2}\;\overline{\eta_3\eta_4} = \overline{\eta_3\eta_4}\;\overline{\eta_1\eta_2}$

Thus, equating the two terms of the two expansions which involve the term $\alpha_1 \cdots \alpha_{2m}$, we get

$$j^{2m}\,\overline{\eta_1 \cdots \eta_{2m}}\,\alpha_1 \cdots \alpha_{2m} = (-1)^m \sum_N \prod_{ij} \overline{\eta_{k_i}\eta_{k_j}}\,\alpha_1 \cdots \alpha_{2m}$$

or
$$\boxed{\overline{\eta_1 \cdots \eta_{2m}} = \sum_N \prod_{ij} \overline{\eta_i \eta_j}}$$

where N is the number of distinct ways of partitioning the η's into products of pairs:

$$\boxed{N = \frac{(2m)!}{2^m m!} = \frac{\cancel{2m}(2m-1)\cancel{(2m-2)}\cdots}{\cancel{2(m)2(m-1)}\cdots} = (2m-1)(2m-3)\cdots 1}$$

If $\eta_1 = \dfrac{f_1}{\sigma_1} = \dfrac{\xi_1 - \overline{\xi}_1}{\sigma_1}$,

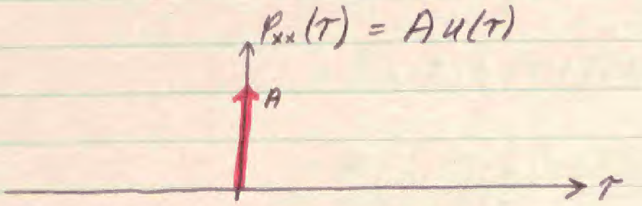
$$\overline{\eta_1 \cdots \eta_n} = \overline{\frac{f_1}{\sigma_1}\cdots\frac{f_n}{\sigma_n}} = \sum\prod \frac{\overline{f_i f_j}}{\sigma_i \sigma_j}$$

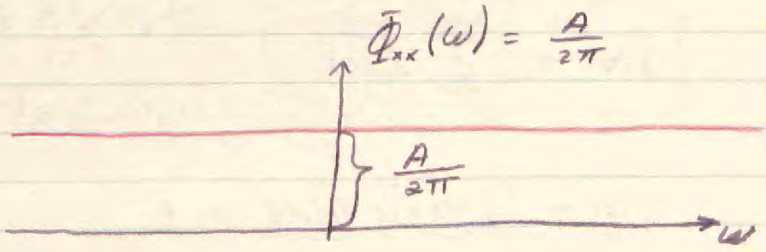
The product $\sigma_1 \cdots \sigma_n$ will occur in each term on both sides of this equation, so we can cancel it out + get

$$\boxed{\overline{f_1 \cdots f_{2m}} = \sum\prod \overline{f_i f_j} \qquad \text{if } \overline{f_i} = 0}$$

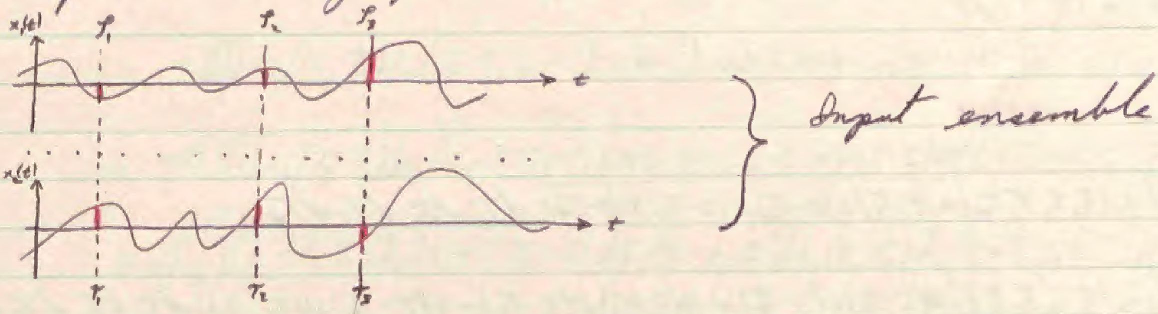
White gaussian noise input to the Volterra model:

The autocorrelation function and power density spectrum for white gaussian noise (w.g.n.) are:





Note that white noise necessarily has zero mean as there is no finite average power at $\omega = 0$.



We can now take an ensemble average at each time t to get $\overline{\rho_1 \rho_2 \cdots \rho_m}$. And we can write

$$\overline{x(t-\tau_1)} = 0$$

$$\overline{x(t-\tau_1)x(t-\tau_2)} = A\,u(\tau_1-\tau_2)$$

$$\overline{x(t-\tau_1)x(t-\tau_2)x(t-\tau_3)} = 0$$

$$\overline{x(t-\tau_1)x(t-\tau_2)x(t-\tau_3)x(t-\tau_4)} = A^2 u(\tau_1-\tau_2)u(\tau_3-\tau_4) + A^2 u(\tau_1-\tau_3)u(\tau_2-\tau_4) + A^2 u(\tau_1-\tau_4)u(\tau_2-\tau_3)$$

etc.

<u>Averages of output for w.g.n. input:</u>

$$\underset{\text{w.g.n.}}{x(t)} \longrightarrow \boxed{\{h_n\}} \longrightarrow y(t) = \sum_{n=1}^{\infty} y_n(t)$$

$$\boxed{\overline{y(t)} = \sum_{n=1}^{\infty} \overline{y_n(t)}}$$

$$\overline{y_1(t)} = \int h_1(\tau_1)\,\overline{x(t-\tau_1)}\,d\tau_1 = 0$$

$$\overline{y_2(t)} = \iint h_2(\tau_1,\tau_2)\,\overline{x(t-\tau_1)x(t-\tau_2)}\,d\tau_1 d\tau_2 = A\iint h_2(\tau_1,\tau_2)\,u(\tau_1-\tau_2)\,d\tau_1\,d\tau_2$$

$$= A\int h_2(\tau_1,\tau_1)\,d\tau_1$$

$$\overline{y_3(t)} = 0$$

$$\overline{y_4(t)} = \iiiint h_4(\tau_1,\tau_2,\tau_3,\tau_4)\,\overline{x(t-\tau_1)x(t-\tau_2)x(t-\tau_3)x(t-\tau_4)}\,d\tau_1\,d\tau_2\,d\tau_3\,d\tau_4$$

$$= A^2\iiiint h_4(\tau_1,\tau_2,\tau_3,\tau_4)\left[u(\tau_1-\tau_2)u(\tau_3-\tau_4) + u(\tau_1-\tau_3)u(\tau_2-\tau_4) + u(\tau_1-\tau_4)u(\tau_2-\tau_3)\right]d\tau_1\,d\tau_2\,d\tau_3\,d\tau_4$$

$$= 3A^2\iint h_4(\tau_1,\tau_1,\tau_2,\tau_2)\,d\tau_1\,d\tau_2$$

In general, $\overline{y_n(t)} = 0$ if n is odd,

$$\boxed{\overline{y_{2m}(t)} = N A^m \int\cdots\int h_{2m}(\tau_1,\tau_1,\tau_2,\tau_2,\ldots,\tau_m,\tau_m)\,d\tau_1\cdots d\tau_m}$$

$$\therefore \quad \overline{y(t)} = \sum_{m=1}^{\infty} \overline{y_{2m}(t)}$$

Autocorrelation of output (w.g.n. input):

$$\overline{y_1(t)\,y_1(t+\tau)} = \iint h_1(\tau_1)\,h_1(\sigma_1)\,\overline{x(t-\tau_1)\,x(t+\tau-\sigma_1)}\,d\tau_1\,d\sigma_1$$

$$= \iint h_1(\tau_1)\,h_1(\tau_2+\tau)\,\overline{x(t-\tau_1)\,x(t-\tau_2)}\,d\tau_1\,d\tau_2 \qquad \text{where } \tau_2 \leftrightarrow \sigma_1 - \tau$$

$$= \iint h_1(\tau_1)\,h_1(\tau_2+\tau)\,A\,u(\tau_2-\tau_1)\,d\tau_1\,d\tau_2$$

$$= A\int h_1(\tau_1)\,h_1(\tau_1+\tau)\,d\tau_1$$

$$\overline{y_2(t)\,y_2(t+\tau)} = \iiiint h_2(\tau_1,\tau_2)\,h_2(\sigma_1,\sigma_2)\,\overline{x(t-\tau_1)\,x(t-\tau_2)\,x(t+\tau-\sigma_1)\,x(t+\tau-\sigma_2)}\,d\tau_1\,d\tau_2\,d\sigma_1\,d\sigma_2$$

$$= \iiiint h_2(\tau_1,\tau_2)\,h_2(\tau_3+\tau,\tau_4+\tau)\,\overline{x(t-\tau_1)\,x(t-\tau_2)\,x(t-\tau_3)\,x(t-\tau_4)}\,d\tau_1\,d\tau_2\,d\tau_3\,d\tau_4$$

$$= A^2\iiiint h_2(\tau_1,\tau_2)\,h_2(\tau_3+\tau,\tau_4+\tau)\left[u(\tau_1-\tau_2)\,u(\tau_3-\tau_4)+u(\tau_1-\tau_3)\,u(\tau_2-\tau_4)+u(\tau_1-\tau_4)\,u(\tau_2-\tau_3)\right]d\tau_1\,d\tau_2\,d\tau_3\,d\tau_4$$

$$= A^2\left[\iint h_2(\tau_1,\tau_1)\,h_2(\tau_3+\tau,\tau_3+\tau)\,d\tau_1\,d\tau_3 + 2\iint h_2(\tau_1,\tau_2)\,h_2(\tau_1+\tau,\tau_2+\tau)\,d\tau_1\,d\tau_2\right]$$

$$= A^2\left[\left\{\int h_2(\tau_1,\tau_1)\,d\tau_1\right\}^2 + 2\iint h_2(\tau_1,\tau_2)\,h_2(\tau_1+\tau,\tau_2+\tau)\,d\tau_1\,d\tau_2\right]$$

$$\overline{y_1(t)\,y_2(t+\tau)} = \iiint h_1(\tau_1)\,h_2(\sigma_1,\sigma_2)\,\overline{x(t-\tau_1)\,x(t+\tau-\sigma_1)\,x(t+\tau-\sigma_2)}\,d\tau_1\,d\sigma_1\,d\sigma_2$$

$$= 0$$

So in general, we see that

$$\overline{y_n(t)\,y_m(t+\tau)} = 0 \quad \text{if } \underline{n+m} \text{ is odd}$$

In the ~~als~~ autocorrelation function

$$\overline{y(t)\,y(t+\tau)} = \sum_n \sum_m \overline{y_n(t)\,y_m(t+\tau)},$$

a typical non-zero cross term is:

$$\overline{y_1(t)\,y_3(t+T)} = \iiiint h_1(\tau_1)\,h_3(\sigma_1,\sigma_2,\sigma_3)\,\overline{x(t-\tau_1)\,x(t+T-\sigma_1)\,x(t+T-\sigma_2)\,x(t+T-\sigma_3)}\,d\tau_1\,d\sigma_1\,d\sigma_2\,d\sigma_3$$

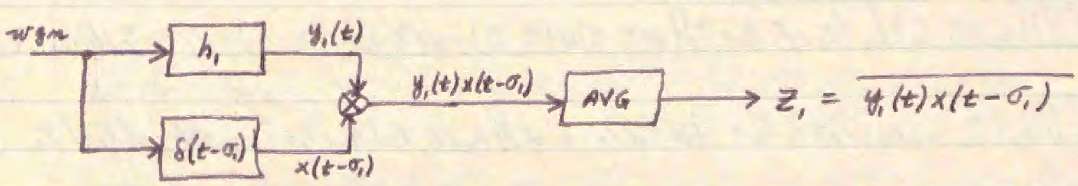
$$= \iiiint h_1(\tau_1)\,h_3(\tau_2+T,\tau_3+T,\tau_4+T)\,\overline{x(t-\tau_1)\,x(t-\tau_2)\,x(t-\tau_3)\,x(t-\tau_4)}\,d\tau_1\,d\tau_2\,d\tau_3\,d\tau_4$$

$$= A^2\iiiint h_1(\tau_1)\,h_3(\tau_2+T,\tau_3+T,\tau_4+T)\Big[u(\tau_1-\tau_2)u(\tau_3-\tau_4)+u(\tau_1-\tau_3)u(\tau_2-\tau_4)+u(\tau_1-\tau_4)u(\tau_2-\tau_3)\Big]d\tau_1\,d\tau_2\,d\tau_3\,d\tau_4$$

$$= \underline{3A^2\iint h_1(\tau_1)\,h_3(\tau_1+T,\tau_3+T,\tau_3+T)\,d\tau_1\,d\tau_3}$$

Measurement of isolated kernels by cross-correlation:

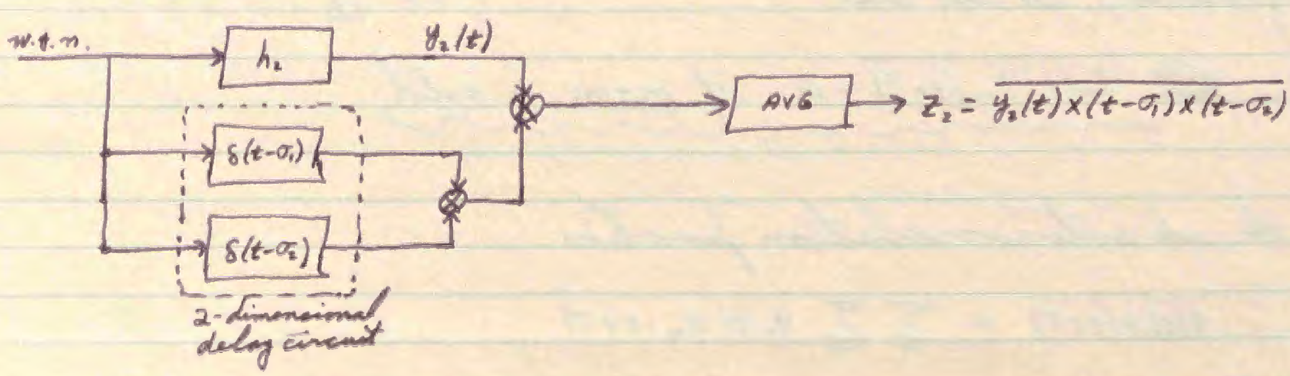
First order:



$$z_1 = \overline{y_1(t)\,x(t-\sigma_1)} = \int h_1(\tau_1)\,\overline{x(t-\tau_1)\,x(t-\sigma_1)}\,d\tau_1 = A\int h_1(\tau_1)\,u(\tau_1-\sigma_1)\,d\tau_1 = A\,h_1(\sigma_1)$$

Thus, $\boxed{h_1(\sigma_1) = \dfrac{z_1}{A}}$

Second order:



$$Z_2 = \iint h_2(\tau_1,\tau_2)\,\overline{x(t-\tau_1)x(t-\tau_2)x(t-\sigma_1)x(t-\sigma_2)}\,d\tau_1\,d\tau_2$$

$$= A^2\iint h_2(\tau_1,\tau_2)\left[u(\tau_1-\tau_2)u(\sigma_1-\sigma_2) + u(\tau_1-\sigma_1)u(\tau_2-\sigma_2) + u(\tau_1-\sigma_2)u(\tau_2-\sigma_1)\right]d\tau_1\,d\tau_2$$

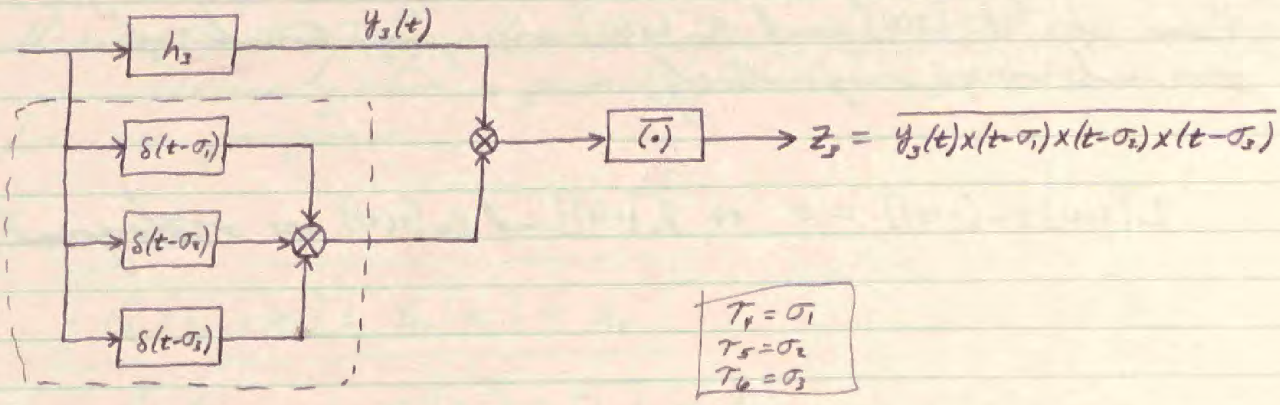
$$= A^2\left[u(\sigma_1-\sigma_2)\int h_2(\tau_1,\tau_1)\,d\tau_1 + h_2(\sigma_1,\sigma_2) + h_2(\sigma_2,\sigma_1)\right]$$

$$= \underline{2A^2 h_2(\sigma_1,\sigma_2) + A^2 u(\sigma_1-\sigma_2)\int h_2(\tau_1,\tau_1)\,d\tau_1}$$

Thus $\boxed{h_2(\sigma_1,\sigma_2) = \dfrac{Z_2}{2A^2}, \quad \sigma_1 \neq \sigma_2}$

Note, however, that if we put a single impulse into this kernel at $t=0$, the output at time σ_1 is $h_2(\sigma_1,\sigma_1)$. Hence in theory we can measure $h_2(\sigma_1,\sigma_2)$ completely, but this will be difficult to do in practice. Especially near $\sigma_1=\sigma_2$, we will see effects of imperfect impulses, not-quite-white noise, etc.

<u>Third order</u>:



$$\boxed{\begin{aligned}\tau_4&=\sigma_1\\ \tau_5&=\sigma_2\\ \tau_6&=\sigma_3\end{aligned}}$$

$$Z_3 = \iiint h_3(\tau_1,\tau_2,\tau_3)\,\overline{x(t-\tau_1)x(t-\tau_2)x(t-\tau_3)x(t-\sigma_1)x(t-\sigma_2)x(t-\sigma_3)}\,d\tau_1\,d\tau_2\,d\tau_3$$

$$= A^3\iiint h_3(\tau_1,\tau_2,\tau_3)\left[u(\tau_1-\tau_2)u(\tau_3-\tau_4)u(\tau_5-\tau_6) + \cdots + u(\tau_1-\tau_4)u(\tau_5-\tau_2)u(\tau_3-\tau_6) + \cdots\right]d\tau_1\,d\tau_2\,d\tau_3$$

$$= A^3\left[6h_3(\tau_4,\tau_5,\tau_6) + 3u(\tau_5-\tau_6)\int h_3(\tau_1,\tau_1,\tau_4)\,d\tau_1 + 3u(\tau_4-\tau_5)\int h_3(\tau_3,\tau_3,\tau_6)\,d\tau_3 + 3u(\tau_6-\tau_4)\int h_3(\tau_2,\tau_2,\tau_5)\,d\tau_2\right]$$

Thus, $\boxed{h_3(\sigma_1,\sigma_2,\sigma_3) = \dfrac{Z_3}{6A^3}, \quad \begin{matrix}\sigma_1\neq\sigma_2\\ \sigma_2\neq\sigma_3\\ \sigma_3\neq\sigma_1\end{matrix}}$ etc. for higher orders.

Orthogonal functionals:

A *homogeneous functional* of degree n is one which has the ~~propter~~ property that

$$\text{if } \mathcal{H}[x(t)] = y_n(t) \text{ then } \mathcal{H}[Kx(t)] = K^n y(t).$$

For example,

$$y_{h_n}(t) = \int\cdots\int h_n(\tau_1,\dots,\tau_n)\, x(t-\tau_1)\cdots x(t-\tau_n)\, d\tau_1 \cdots d\tau_n$$

is a homogeneous functional. When $x(t) \to Kx(t)$, we get $K^n y_{h_n}(t)$.

A *non-homogeneous functional* is a functional which does not possess this property; i.e., one that is not homogeneous.

~~Orthog~~ Two functionals are said to be orthogonal if the ~~two~~ average value of their product is zero.

Thus, if $g_n[x(t)]$ and $g_m[x(t)]$ are functionals of order n and m, respectively,

$$\overline{g_n[x(t)]\, g_m[x(t)]} = 0 \Leftrightarrow g_n[x(t)] \text{ and } g_m[x(t)] \text{ are orthogonal.}$$

Orthogonal functional description of a system:

From the set, $\{y_{h_n}(t)\}$, of homogeneous functionals of the form $\int\cdots\int h_n(\tau_1,\ldots,\tau_n)\,x(t-\tau_1)\cdots x(t-\tau_n)\,d\tau_1\cdots d\tau_n = y_{h_n}(t)$, we want to determine a set, $\{G_n[h_n,\ldots,h_0,x(t)]\}$, of orthogonal functionals which can be used to ~~des~~ describe system behavior. These functionals will not necessarily be homogeneous.

We will assume that we are dealing with white gaussian noise inputs, $x(t) = $ w.g.n.

The functionals will be of the form:

$$G_n[h_n,\ldots,h_0,x(t)] = \sum_{k=1}^{n}\int\cdots\int h_k(\tau_1,\ldots,\tau_k)\,x(t-\tau_1)\cdots x(t-\tau_k)\,d\tau_1\cdots d\tau_k + h_0$$

where h_0 is a constant.

These kernels ~~h~~ are not the same as those occurring in the Volterra formulation.

We will choose each functional to be orthogonal to all homogeneous functionals of lower degree, including all constants. They will, therefore, be orthogonal to each other.

Normalization is possible, but is not necessary for our purposes.

Zero order

$$\boxed{G_0[h_0,x(t)] = G_0[h_0] = h_0}$$

Normalization: $1 = \overline{G_0[h_0]\,G_0[h_0]} = h_0^2 \Rightarrow \underline{h_0^2 = 1}$

First order G-functional:

Let $g_1[h_1, h_0, x(t)] = \int h_1(\tau_1)x(t-\tau_1)d\tau_1 + h_0 = y_{h_1}(t) + y_{h_0}$; h_1, h_0 arbitrary

Then $G_1[h_1, h_0, x(t)]$ will be that particular $g_1[h_1, h_0, x(t)]$ for which the following orthogonality condition holds:

$$0 = \overline{G_1[h_1, h_0, x(t)]\, k_0} = k_0 \int h_1(\tau_1)\overline{x(t-\tau_1)}d\tau_1 + k_0 h_0 = k_0 h_0$$

$$\Rightarrow \underline{h_0 = 0}$$

Thus, $\boxed{G_1[h_1, h_0, x(t)] = G_1[h_1, x(t)] = \int h_1(\tau_1)x(t-\tau_1)d\tau_1}$

Normalization: $1 = \overline{G_1^2[h_1, x(t)]} = \iint h_1(\tau_1)h_1(\tau_2)\overline{x(t-\tau_1)x(t-\tau_2)}\,d\tau_1 d\tau_2$

$$\boxed{1 = A\int h_1^2(\tau_1)d\tau_1}$$

Second-order ~~for~~ G-functional:

$G_2[h_2, h_1, h_0, x(t)]$ will be that functional of the form
$g_2[h_2, h_1, h_0, x(t)] = y_{h_2}(t) + y_{h_1}(t) + y_{h_0}$ which satisfies the orthogonality ~~condi~~ requirements:

(1) $0 = \overline{k_0 G_2[h_2, h_1, h_0, x(t)]}$; k_0 is any constant

(2) $0 = \overline{G_2[h_2, h_1, h_0, x(t)]\, y_{k_1}(t)}$; $y_{k_1}(t) = \int k_1(\tau_1)x(t-\tau_1)d\tau_1$; k_1 is any 1st order kernel.

Requirement (1) becomes:

$$0 = \overline{k_0[y_{h_2}(t) + y_{h_1}(t) + y_{h_0}]}$$

Now we know that terms involving an odd number of ~~b~~ products of $x(t-\tau_i)$ will be zero when we average, so we can write

$$0 = \overline{k_0 y_{h_2}(t)} + k_0 h_0 = k_0\left[\iint h_2(\tau_1,\tau_2)\overline{x(t-\tau_1)x(t-\tau_2)}\,d\tau_1 d\tau_2 + h_0\right]$$

$0 = k_0 \left[A \int h_2(\tau_1, \tau_1)\, d\tau_1 + h_0 \right]$ for any constant k_0

$\Rightarrow \boxed{h_0 = -A \int h_2(\tau_1, \tau_1)\, d\tau_1}$

Requirement (2) becomes

$$0 = \overline{y_{k_1}(t)\left[y_{h_2}(t) + y_{h_1}(t) + y_{h_0} \right]} = \overline{y_{k_1}(t)\, y_{h_2}(t)}^{\,0} + \overline{y_{k_1}(t)\, y_{h_1}(t)} + \overline{y_{k_1}(t)\, y_{h_0}}^{\,0}$$

$\cancel{\xi[\iiint h_2(\tau_1,\tau_2)\,k_1(\tau_3)\,x\,t}$ [Odd terms cancel because $x(t)$ is gaussian]

$$0 = \iint h_1(\tau_1)\, k_1(\tau_2)\, \overline{x(t-\tau_1)\, x(t-\tau_2)}\, d\tau_1\, d\tau_2 = A \int h_1(\tau_1)\, k_1(\tau_1)\, d\tau_1$$

But $k_1(\tau_1)$ is arbitrary, so we must have

$$\boxed{h_1(\tau_1) = 0}$$

Thus, $G_2[h_2, h_1, h_0, x(t)]$ depends only on h_2 because $h_1 = 0$ and h_0 can be found from h_2. So:

$$\boxed{G_2[h_2, h_1, h_0, x(t)] = G_2[h_2, x(t)] = \iint h_2(\tau_1, \tau_2)\, x(t-\tau_1)\, x(t-\tau_2)\, d\tau_1\, d\tau_2 - A \int h_2(\tau_1, \tau_1)\, d\tau_1}$$

Normalization: $\overline{G_2^2[h_2, x(t)]} = 0 \Rightarrow 1 = \overline{y_{h_2}^2(t)} - 2A \int h_2(\tau_1, \tau_1)\, d\tau_1\, \overline{y_{h_2}(t)} + A^2 \left[\int h_2(\tau, \tau)\, d\tau \right]^2 =$

$$1 = \iiiint h_2(\tau_1, \tau_2)\, h_2(\tau_3, \tau_4)\, \overline{x(t-\tau_1)\, x(t-\tau_2)\, x(t-\tau_3)\, x(t-\tau_4)}\, d\tau_1\, d\tau_2\, d\tau_3\, d\tau_4$$

$$- 2A \int h_2(\tau_1, \tau_1)\, d\tau_1 \iint h_2(\tau_2, \tau_3)\, \overline{x(t-\tau_2)\, x(t-\tau_3)}\, d\tau_2\, d\tau_3 + A^2 \left[\int h_2(\tau_1, \tau_1)\, d\tau_1 \right]^2$$

$$1 = \underbrace{A^2 \left[\int h_2(\tau_1, \tau_1)\, d\tau_1 \right]^2 + 2A^2 \iint h_2^2(\tau_1, \tau_2)\, d\tau_1\, d\tau_2}_{\text{from page 25, } \overline{y_2(t)\, y_2(t+0)}} - 2A^2 \left[\int h_2(\tau_1, \tau_1)\, d\tau_1 \right]^2 + A^2 \left[\int h_2(\tau_1, \tau_1)\, d\tau_1 \right]^2$$

$$\boxed{1 = 2A^2 \iint h_2^2(\tau_1, \tau_2)\, d\tau_1\, d\tau_2}$$

Third order G-functional:

$G_3[h_3, h_2, h_1, h_0, x(t)]$ will be that particular $g_3[h_3, h_2, h_1, h_0, x(t)] = y_{h_3}(t) + y_{h_2}(t) + y_{h_1}(t) + y_{h_0}$ which satisfies:

$$\left.\begin{array}{ll} (1) & \overline{G_3[h_3, h_2, h_1, h_0, x(t)]\, K_0} = 0 \\ (2) & \overline{G_3[h_3, h_2, h_1, h_0, x(t)]\, y_{K_1}(t)} = 0 \\ (3) & \overline{G_3[h_3, h_2, h_1, h_0, x(t)]\, y_{K_2}(t)} = 0 \end{array}\right\}$$

K_0, K_1, K_2 arbitrary kernels of functionals of order, 0, 1, 2.

These requirements become

(1) $0 = \overline{[y_{h_3}(t) + y_{h_2}(t) + y_{h_1}(t) + h_0]\, K_0} = K_0\left[\cancel{\overline{y_{h_3}(t)}}^{\,0} + \overline{y_{h_2}(t)} + \cancel{\overline{y_{h_1}(t)}}^{\,0} + h_0\right]$

$$0 = K_0\left[\iint h_2(\tau_1, \tau_2)\, \overline{x(t-\tau_1)\, x(t-\tau_2)}\, d\tau_1\, d\tau_2 + h_0\right]$$

$$0 = h_0 + A\int h_2(\tau_1, \tau_1)\, d\tau_1 \quad \Rightarrow \quad \boxed{h_0 = -A\int h_2(\tau_1, \tau_1)\, d\tau_1}$$

(2) $0 = \overline{y_{K_1}(t)\,[y_{h_3}(t) + \cancel{y_{h_2}(t)} + y_{h_1}(t) + \cancel{h_0}]} = \overline{y_{K_1}(t)\, y_{h_3}(t)} + \overline{y_{K_1}(t)\, y_{h_1}(t)}$

$$0 = \iiiint h_3(\tau_1, \tau_2, \tau_3)\, k_1(\tau_4)\, \overline{x(t-\tau_1)\, x(t-\tau_2)\, x(t-\tau_3)\, x(t-\tau_4)}\, d\tau_1\, d\tau_2\, d\tau_3\, d\tau_4 + \iint h_1(\tau_1)\, k_1(\tau_2)\, \overline{x(t-\tau_1)\, x(t-\tau_2)}\, d\tau_1\, d\tau_2$$

$$0 = \underbrace{3A^2 \iint h_3(\tau_2, \tau_2, \tau_1)\, k_1(\tau_1)\, d\tau_1\, d\tau_2}_{\text{from page 26, } \overline{y_1(t)\, y_3(t+\sigma)}} + A\int h_1(\tau_1)\, k_1(\tau_1)\, d\tau_1$$

$$0 = A\int k_1(\tau_1)\, d\tau_1 \left[3A\int h_3(\tau_2, \tau_2, \tau_1)\, d\tau_2 + h_1(\tau_1)\right] = 0 \quad , \; k_1(\cdot) \text{ arbitrary.}$$

$$\Rightarrow \quad 0 = 3A\int h_3(\tau_2, \tau_2, \tau_1)\, d\tau_2 + h_1(\tau_1)$$

$$\Rightarrow \quad \boxed{h_1(\tau_1) = -3A\int h_3(\tau_2, \tau_2, \tau_1)\, d\tau_2}$$

$$(3)\quad 0 = \overline{y_{k_2}(t)\left[\cancel{y_{h_3}(t)} + y_{h_2}(t) + \cancel{y_{h_1}(t)} + h_0\right]} = \overline{y_{k_2}(t)\,y_{h_2}(t)} + h_0\,\overline{y_{k_2}(t)}$$

$$0 = \iiiint h_2(\tau_1,\tau_2)k_2(\tau_3,\tau_4)\overline{x(t-\tau_1)x(t-\tau_2)x(t-\tau_3)x(t-\tau_4)}\,d\tau_1 d\tau_2 d\tau_3 d\tau_4 + h_0\iint k_2(\tau_1,\tau_2)\overline{x(t-\tau_1)x(t-\tau_2)}\,d\tau_1 d\tau_2$$

$$= A^2\iint h_2(\tau_1,\tau_1)k_2(\tau_2,\tau_2)\,d\tau_1 d\tau_2 + 2A^2\iint h_2(\tau_1,\tau_2)k_2(\tau_1,\tau_2)\,d\tau_1 d\tau_2 + Ah_0\int k_2(\tau_1,\tau_1)\,d\tau_1$$

But $h_0 = -A\int h_2(\tau_1,\tau_1)\,d\tau_1$, so

$$0 = A^2\left[\int h_2(\tau_1,\tau_1)d\tau_1\right]\left[\int k_2(\tau_1,\tau_1)d\tau_1\right] + 2A^2\iint h_2(\tau_1,\tau_2)k_2(\tau_1,\tau_2)\,d\tau_1 d\tau_2 - A^2\int k_2(\tau_1,\tau_1)d\tau_1\int h_2(\tau_2,\tau_2)d\tau_2$$

$$0 = A^2\iint h_2(\tau_1,\tau_2)k_2(\tau_1,\tau_2)\,d\tau_1 d\tau_2 \quad , \quad k_2(\tau_1,\tau_2) \text{ arbitrary}$$

$$\Rightarrow \boxed{h_2 = 0 \;\Rightarrow\; h_0 = 0}$$

So

$$G_3[h_3,h_2,h_1,h_0,x(t)] = G_3[h_3,x(t)] = \iiint h_3(\tau_1,\tau_2,\tau_3)x(t-\tau_1)x(t-\tau_2)x(t-\tau_3)\,d\tau_1 d\tau_2 d\tau_3 - \int h_1(\tau_1)x(t-\tau_1)\,d\tau_1$$

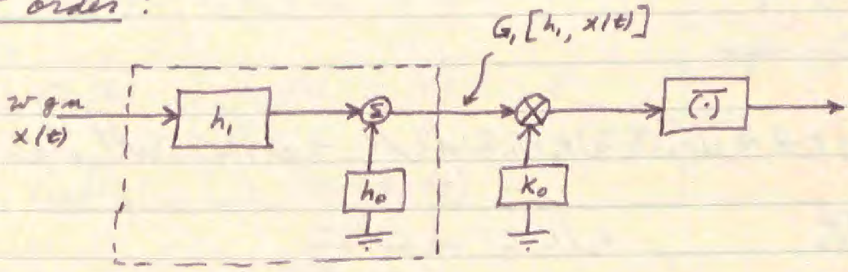
or

$$\boxed{G_3[h_3,x(t)] = y_{h_3}(t) - 3A\iint h_3(\tau_2,\tau_2,\tau_1)x(t-\tau_1)\,d\tau_1 d\tau_2}$$

Orthogonality of G-functions:

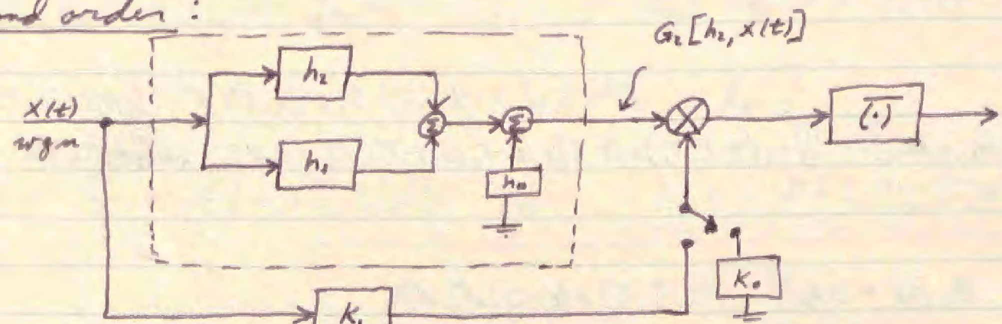
In the following diagrams, $\{h_n\}$ refers to the system kernels, while $\{k_n\}$ are arbitrary kernels.

First order:



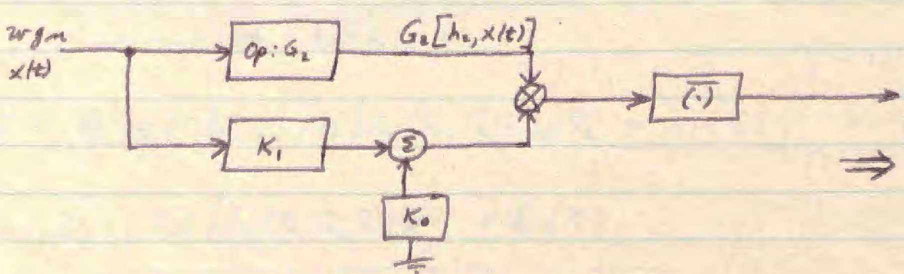
$$\overline{G_1[h_1, x(t)]\, k_0} = 0$$

Second order:



$$\overline{G_2[h_2, x(t)]\, k_0} = 0$$
$$\overline{G_2[h_2, x(t)]\, y_{k_1}(t)} = 0$$

As a result, $G_2[h_2, x(t)]$ is orthogonal to $G_1[h_1, x(t)]$:



$$\overline{G_2[h_2, x(t)]\{y_{k_1}(t) + k_0\}} = 0$$
$$\Rightarrow \overline{G_2[h_2, x(t)]\, G_1[h_1, x(t)]} = 0$$

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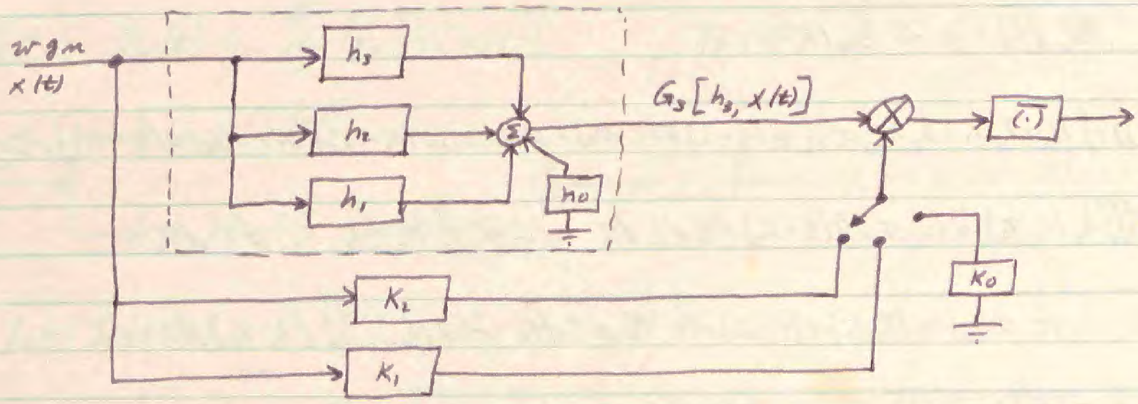
Third order:



$$\overline{G_3[h_3, x(t)]\, K_0} = 0$$
$$\overline{G_3[h_3, x(t)]\, y_{K_1}(t)} = 0$$
$$\overline{G_3[h_3, x(t)]\, y_{K_2}(t)} = 0$$

Mean square value of $G_n[h_n, x(t)]$:

Let us separate out the first term of $G_n[h_n, x(t)]$, so we have

$$G_n[h_n, x(t)] = y_{h_n}(t) + R \quad ; \quad R \text{ is a functional of degree} < n.$$

Then

$$\overline{G_n^2[h_n, x(t)]} = \overline{G_n[h_n, x(t)]\{y_{h_n}(t) + R\}}$$

$$= \overline{G_n[h_n, x(t)]\, y_{h_n}(t)} + \underbrace{\overline{G_n[h_n, x(t)]\, R}}_{0}$$

But R is of degree less than n, so the second term is zero. Our problem is now to evaluate

$$\overline{G_n^2[h_n, x(t)]} = \overline{[y_{h_n}(t) + R]\, y_{h_n}(t)}$$

The general result will not be proven here, but it comes out

$$\overline{[y_{h_n}(t) + R]\, y_{h_n}(t)} = \overline{[y_{h_n}(t) + R][\text{functionals of degree} < n]} + n!\, A^n \int \cdots \int h_n^2(\tau_1, \ldots, \tau_n)\, d\tau_1 \cdots d\tau_n$$

so

$$\boxed{\overline{G_n^2[h_n, x(t)]} = n!\, A^n \int \cdots \int h_n^2(\tau_1, \ldots, \tau_n)\, d\tau_1 \cdots d\tau_n}$$

Examples; $n = 2$:

$$\overline{G_2^2[h_2, x(t)]} = \overline{y_{h_2}^2(t)} + \overline{R\, y_{h_2}(t)}$$

(1) $$\overline{y_{h_2}^2(t)} = A^2 \iiiint h_2(\tau_1,\tau_2)\, h_2(\sigma_1,\sigma_2)\big[u(\tau_1-\tau_2)u(\sigma_1-\sigma_2) + u(\tau_1-\sigma_1)u(\tau_2-\sigma_2) + u(\tau_1-\sigma_2)u(\tau_2-\sigma_1)\big]\, d\tau_1 d\tau_2\, d\sigma_1 d\sigma_2$$

$$= A^2 \iiint h_2(\tau_1,\tau_2)\, h_2(\sigma_1,\sigma_1)\, u(\tau_1-\tau_2)\, d\tau_1 d\tau_2 d\sigma_1 + 2A^2 \iint h_2^2(\tau_1,\tau_2)\, d\tau_1 d\tau_2$$

$$= A \int d\sigma_1\, h_2(\sigma_1,\sigma_1) \overline{\iint h_2(\tau_1,\tau_2)\, x(t-\tau_1)\, x(t-\tau_2)\, d\tau_1 d\tau_2} + 2A^2 \iint h_2^2(\tau_1,\tau_2)\, d\tau_1 d\tau_2$$

$$\cancel{A\,[\text{constant}]\, \overline{y_{h_2}(t)} + 2A^2 \iint h_2^2(\tau_1,\tau_2)\, d\tau_1 d\tau_2}$$

(2) $$\overline{y_{h_2}(t)\, R} = \overline{y_{h_2}(t)\, h_0} \quad \cancel{-2A\, y_{h_2}(t) \int h_2(\tau_1,\tau_1)\, d\tau_1}$$

$$= \overline{\left(-A \int h_2(\tau_1,\tau_1)\, d\tau_1\right) y_{h_2}(t)}$$

So $$G_2^2[h_2, x(t)] = (1) + (2) = A \int h_2(\sigma_1,\sigma_1)\, d\sigma_1\, \overline{y_{h_2}(t)} + 2A^2 \iint h_2^2(\tau_1,\tau_2)\, d\tau_1 d\tau_2 - A \int h_2^2(\tau_1,\tau_1)\, d\tau_1\, y_{h_2}(t)$$

$$G_2^2[h_2, x(t)] = 2A^2 \iint h_2^2(\tau$$

(2) $$\overline{y_{h_2}(t)\, R} = -A \int h_2(\tau_1,\tau_1)\, d\tau_1\, \overline{y_{h_2}(t)}$$

$$= -A^2 \int h_2(\tau_1,\tau_1)\, d\tau_1 \iint h_2(\sigma_1,\sigma_2)\, u(\sigma_1-\sigma_2)\, d\sigma_1 d\sigma_2$$

$$= -A^2 \int h_2(\tau_1,\tau_1)\, d\tau_1 \int h_2(\sigma_1,\sigma_1)\, d\sigma_1$$

So $$\overline{G_2^2[h_2, x(t)]} = \overline{A \int h_2(\sigma_1,\sigma_1)\, d\sigma_1 \left[\iint h_2(\tau_1,\tau_2)\, x(t-\tau_1)\, x(t-\tau_2)\, d\tau_1 d\tau_2 - A \int h_2(\tau_1,\tau_1)\, d\tau_1\right]}$$

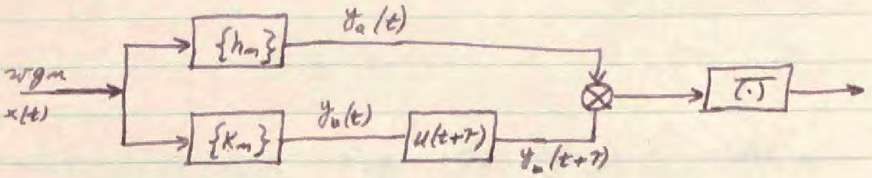
$$+ 2A^2 \iint h_2^2(\tau_1,\tau_2)\, d\tau_1 d\tau_2$$

$$= [\text{constant}]\, \overline{G_2[h_2, x(t)]} + 2A^2 \iint h_2^2(\tau_1,\tau_2)\, d\tau_1 d\tau_2$$

$$= 2A^2 \iint h_2^2(\tau_1,\tau_2)\, d\tau_1 d\tau_2$$

Cross-correlation via G-functions:

Let $\quad y_a(t) = \sum_{0}^{\infty} G_n[h_n, x(t)] \quad , \quad y_b(t) = \sum_{0}^{\infty} G_n[k_n, x(t)]$



$$\overline{y_a(t)\, y_b(t+\tau)} = \varphi_{ab}(\tau)$$

$$\overline{y_a(t)\, y_b(t+\tau)} = \sum_n \sum_m \overline{G_n[h_n, x(t)]\, G_m[k_m, x(t+\tau)]}$$

$$= \sum_n \overline{G_n[h_n, x(t)]\, G_n[k_n, x(t+\tau)]} \,.$$

$$= \sum_n \overline{[y_{h_n}(t) + R_{an}]\, y_{k_n}(t+\tau)} \quad ; \; y_{k_n}(t) = \int\cdots\int k_n(\tau_1, \ldots, \tau_n)\, x(t-\tau_1)\cdots x(t-\tau_n)\, d\tau_1\cdots d\tau_n$$

$$= \sum_n \overline{[y_{h_n}(t) + R_{an}] \int\cdots\int k_n(\sigma_1+\tau, \ldots, \sigma_n+\tau)\, x(t-\sigma_1)\cdots x(t-\sigma_n)\, d\sigma_1\cdots d\sigma_n}$$

where $\tau - \tau_k = -\sigma_k \quad , \; k = 1, \ldots, n$

But from the result quoted on page 35, we see that this is

$$\boxed{\overline{y_a(t)\, y_b(t+\tau)} = \sum_{0}^{\infty} n!\, A^n \int\cdots\int h_n(\tau_1, \ldots, \tau_n)\, k_n(\tau_1+\tau, \ldots, \tau_n+\tau)\, d\tau_1\cdots d\tau_n} \quad \boxed{= \varphi_{ab}(\tau)}$$

As special cases of this, we see

$$\boxed{\varphi_{aa}(\tau) = \overline{y_a(t)\, y_a(t+\tau)} = \sum_{0}^{\infty} n!\, A^n \int\cdots\int h_n(\tau_1, \ldots, \tau_n)\, h_n(\tau_1+\tau, \ldots, \tau_n+\tau)\, d\tau_1\cdots d\tau_n}$$

and

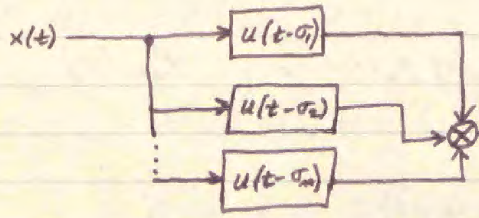
$$\boxed{\overline{y_a^2(t)} = \sum_{0}^{\infty} n!\, A^n \int\cdots\int h_n^2(\tau_1, \ldots, \tau_n)\, d\tau_1\cdots d\tau_n}$$

cf. p. 35.

$$\boxed{\overline{G_n[h_n, x(t)]\, G_n[k_n, x(t+\tau)]} = n!\, A^n \int\cdots\int h_n(\tau_1, \ldots, \tau_n)\, k_n(\tau_1+\tau, \ldots, \tau_n+\tau)\, d\tau_1\cdots d\tau_n}$$

Measurement of kernels by ~~cross~~ auto-correlation: c.f. p 43.

Multi-dimensional delay lines:



$$y(t) = y_n(t, \sigma_1, \ldots, \sigma_n) = x(t-\sigma_1)\,x(t-\sigma_2) \cdots x(t-\sigma_n)$$

The output of a multi-dimensional delay line is a homogeneous functional:

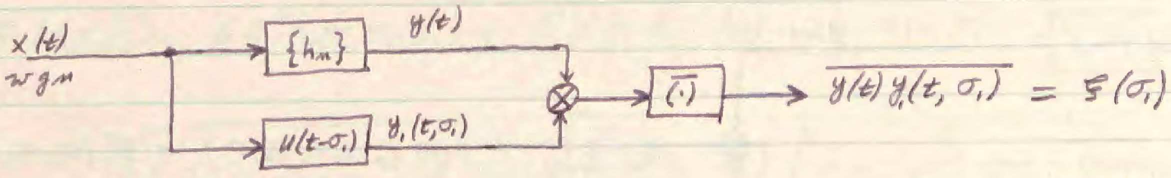
$$\boxed{\begin{array}{l} y_n(t, \sigma_1, \sigma_2, \ldots, \sigma_n) = x(t-\sigma_1) \cdots x(t-\sigma_n) \\ \leftrightarrow \text{a functional of degree } n \end{array}}$$

Note that this delay line has a kernel

$$\cancel{h_n} \quad h_n(\tau_1, \cdots, \tau_n) = u(\tau_1 - \sigma_1) \cdots u(\tau_n - \sigma_n)$$

which is not ~~have~~ symmetrical.

Calculation of $h_1(\tau_1)$:



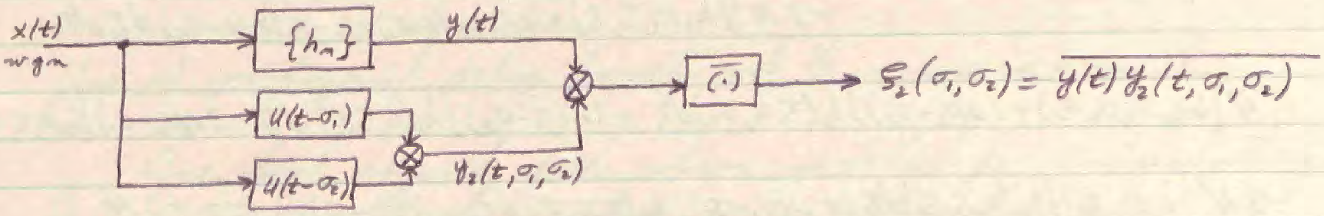
$$\overline{y(t)\,y_1(t,\sigma_1)} = \sum_{0}^{\infty} \overline{G_m[h_m, x(t)]\,x(t-\sigma_1)}$$

$$= \overline{G_1[h_1, x(t)]\,x(t-\sigma_1)} \qquad \text{since } x(t-\sigma_1) \text{ is a functional of degree 1.}$$

$$= \int h_1(\tau_1)\,\overline{x(t-\tau_1)\,x(t-\sigma_1)}\,d\tau_1 = A\int h_1(\tau_1)\,u(\tau_1-\sigma_1)\,d\sigma_1$$

$$\boxed{\overline{y(t)\,y_1(t,\sigma_1)} = A\,h_1(\sigma_1)} \quad \text{or} \quad \boxed{h_1(\sigma_1) = \frac{\xi(\sigma_1)}{A} = \frac{\overline{y(t)\,y_1(t,\sigma_1)}}{A}}$$

Calculation of $h_2(\tau_1,\tau_2)$:



$$\xi_2(\sigma_1,\sigma_2) = \sum_{0}^{\infty} \overline{G_m[h_m, x(t)]\,x(t-\sigma_1)\,x(t-\sigma_2)}$$

$$= \overline{G_1[h_1, x(t)]\,x(t-\sigma_1)\,x(t-\sigma_2)} + \overline{G_2[h_2, x(t)]\,x(t-\sigma_1)\,x(t-\sigma_2)}$$

$$= \int h_1(\tau_1)\,\underbrace{\overline{x(t-\tau_1)\,x(t-\sigma_1)\,x(t-\sigma_2)}}_{0}\,d\tau_1$$

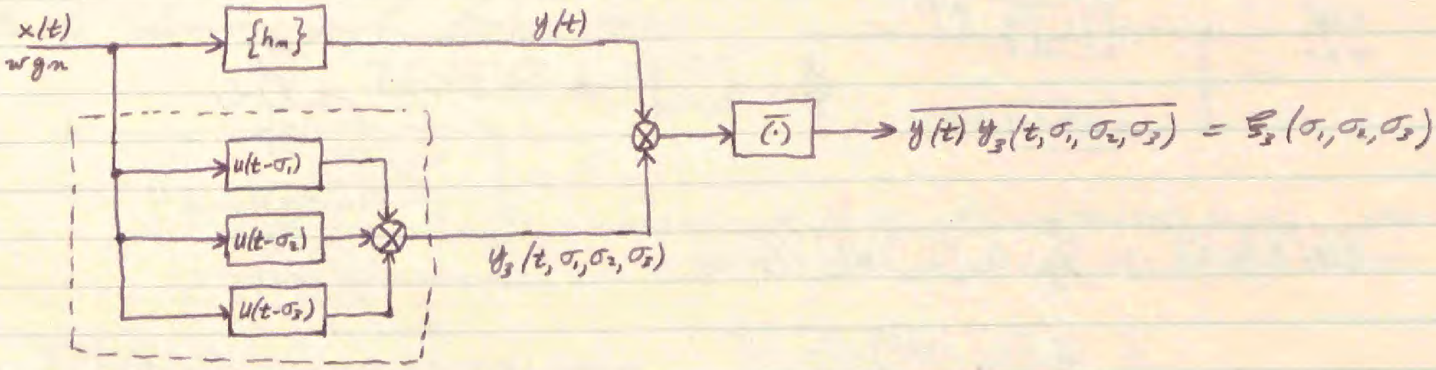
$$+ \iint h_2(\tau_1,\tau_2)\,\overline{x(t-\tau_1)\,x(t-\tau_2)\,x(t-\sigma_1)\,x(t-\sigma_2)}\,d\tau_1 d\tau_2 - A\int h_2(\tau_1,\tau_1)\,d\tau_1\,\overline{x(t-\sigma_1)\,x(t-\sigma_2)}$$

$$= A^2\left[u(\sigma_1-\sigma_2)\int h_2(\tau_1,\tau_1)\,d\tau_1 + h_2(\sigma_1,\sigma_2) + h_2(\sigma_2,\sigma_1)\right] - A^2 u(\sigma_1-\sigma_2)\int h_2(\tau_1,\tau_1)\,d\tau_1$$

$$= 2A^2\,h_2(\sigma_1,\sigma_2)$$

so $$\boxed{h_2(\sigma_1,\sigma_2) = \frac{\xi_2(\sigma_1,\sigma_2)}{2A^2} = \frac{\overline{y(t)\,y_2(t,\sigma_1,\sigma_2)}}{2A^2}}$$

Calculation of $h_3(\tau_1, \tau_2, \tau_3)$:



$y_3(t, \sigma_1, \sigma_2, \sigma_3)$ is a kernel of degree three :

$$\xi_3(\sigma_1,\sigma_2,\sigma_3) = \overline{\left(G_0 + G_1[h_1, x(t)] + G_2[h_2, x(t)] + G_3[h_3, x(t)]\right) x(t-\sigma_1)\,x(t-\sigma_2)\,x(t-\sigma_3)}$$

Going term by term, we get :

$$\overline{G_3[h_3, x(t)]\,x(t-\sigma_1)\,x(t-\sigma_2)\,x(t-\sigma_3)} = \iiint h_3(\tau_1,\tau_2,\tau_3)\,\overline{x(t-\sigma_1)x(t-\sigma_2)x(t-\sigma_3)x(t-\tau_1)x(t-\tau_2)x(t-\tau_3)}\,d\tau_1 d\tau_2 d\tau_3$$

$$-3A\iint h_3(\tau_1,\tau_2,\tau_2)\,\overline{x(t-\tau_1)x(t-\sigma_1)x(t-\sigma_2)x(t-\sigma_3)}\,d\tau_1 d\tau_2$$

$$= A^3\Big[6h_3(\sigma_1,\sigma_2,\sigma_3) + 3u(\sigma_2-\sigma_3)\int h_3(\tau_1,\tau_1,\sigma_1)\,d\tau_1 + 3u(\sigma_1-\sigma_3)\int h_3(\tau_3,\tau_3,\sigma_2)\,d\tau_3 + 3u(\sigma_2-\sigma_1)\int h_3(\tau_2,\tau_2,\sigma_3)\,d\tau_2\Big]$$

$$-3A^3\Big[u(\sigma_1-\sigma_2)\int h_3(\tau_1,\tau_1,\sigma_3)\,d\tau_1 + u(\sigma_1-\sigma_3)\int h_3(\tau_1,\tau_1,\sigma_2)\,d\tau_1 + u(\sigma_2-\sigma_3)\int h_3(\tau_1,\tau_1,\sigma_1)\,d\tau_1\Big]$$

$$= 6A^3 h_3(\sigma_1,\sigma_2,\sigma_3)$$

$$\overline{G_2[h_2, x(t)]\,x(t-\sigma_1)x(t-\sigma_2)x(t-\sigma_3)} = \underbrace{\overline{y_{h_2}(t)\,x(t-\sigma_1)x(t-\sigma_2)x(t-\sigma_3)}}_{0} - A\underbrace{\int h_2(\tau_1,\tau_1)\,d\tau_1}_{0}\,\overline{x(t-\sigma_1)x(t-\sigma_2)x(t-\sigma_3)} = 0.$$

$$\overline{G_1[h_1, x(t)]\,x(t-\sigma_1)x(t-\sigma_2)x(t-\sigma_3)} = \int h_1(\tau_1)\,\overline{x(t-\tau_1)x(t-\sigma_1)x(t-\sigma_2)x(t-\sigma_3)}\,d\tau_1$$

$$= A^2\left[u(\sigma_2-\sigma_3)h_1(\sigma_1) + u(\sigma_1-\sigma_2)h_1(\sigma_3) + u(\sigma_1-\sigma_3)\right]$$

$$G_0\,\overline{x(t-\sigma_1)\,x(t-\sigma_2)\,x(t-\sigma_3)} = 0.$$

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So,

$$\xi_3(\sigma_1,\sigma_2,\sigma_3) = 6A^3 h_3(\sigma_1,\sigma_2,\sigma_3) + A^2\left[u(\sigma_1-\sigma_2)\,h_1(\sigma_3) + u(\sigma_1-\sigma_3)\,h_1(\sigma_2) + u(\sigma_2-\sigma_3)\,h_1(\sigma_1)\right]$$

or

$$\boxed{h_3(\sigma_1,\sigma_2,\sigma_3) = \frac{1}{6A^3}\,\overline{y(t)\,y_3(t,\sigma_1,\sigma_2,\sigma_3)}, \quad \sigma_1\neq\sigma_2\neq\sigma_3\neq\sigma_1},$$

Advantage of G-functional representation:

Notice that by using an orthogonal functional representation for the system, we are able to measure each of the kernels directly. In the Volterra representation, we could not do this, and could measure ~~the~~ only isolated kernels of that expansion. (Note again that the kernels of the functionals used to represent the system differ with the representation chosen.)

Use of orthogonal delay functionals to measure kernels:

One bad aspect of the above method for calculating $h_3(\sigma_1,\sigma_2,\sigma_3)$ is that the expression is not valid when the arguments are pair-wise equal. Another way to calculate this kernel is to use the kernel of

$$y_3(t,\sigma_1,\sigma_2,\sigma_3) = \iiint u(\tau_1-\sigma_1)\,u(\tau_2-\sigma_2)\,u(\tau_3-\sigma_3)\,x(t-\tau_1)\,x(t-\tau_2)\,x(t-\tau_3)\,d\tau_1\,d\tau_2\,d\tau_3$$

$$= \iiint K_3(\tau_1,\tau_2,\tau_3)\,x(t-\tau_1)\,x(t-\tau_2)\,x(t-\tau_3)\,d\tau_1\,d\tau_2\,d\tau_3$$

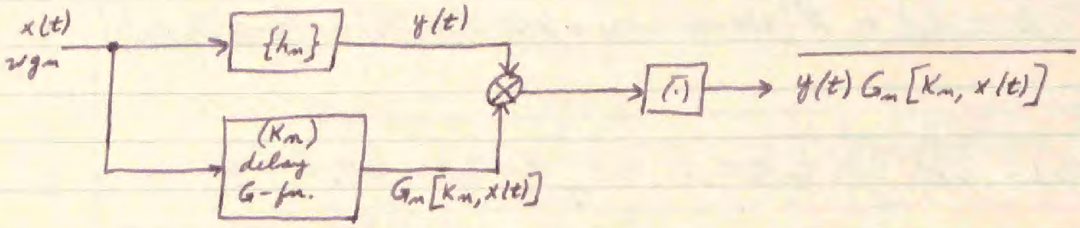
where $K_3(\tau_1,\tau_2,\tau_3) = u(\tau_1-\sigma_1)\,u(\tau_2-\sigma_2)\,u(\tau_3-\sigma_3)$. , ~~unsymmetric~~.

We can use this kernel to form a 3rd order G-functional:

$G_3[K_3, x(t)]$, or in general, $\underbrace{K_n(\tau_1,\ldots,\tau_n) = u(\tau_1-\sigma_1)\cdots u(\tau_n-\sigma_n)}$

which can be used to give $\underbrace{G_n[K_n, x(t)]}$.

We can now connect our system as follows:



$$\overline{y(t)\, G_n[K_n, x(t)]} = \sum_{m=0}^{\infty} \overline{G_m[h_m, x(t)]\, G_n[K_n, x(t)]} = \overline{G_n[h_n, x(t)]\, G_n[K_n, x(t)]}$$

$$= n!\, A^n \int \cdots \int h_n(\tau_1, \dots, \tau_n)\, K_n(\tau_1, \dots, \tau_n)\, d\tau_1 \cdots d\tau_n \quad \text{from page 37 } (\tau = 0).$$

$$= n!\, A^n \int \cdots \int h_n(\tau_1, \dots, \tau_n)\, u(\tau_1 - \sigma_1) \cdots u(\tau_n - \sigma_n)\, d\tau_1 \cdots d\tau_n$$

$$= n!\, A^n\, h_n(\sigma_1, \dots, \sigma_n)$$

or

$$\boxed{h_n(\sigma_1, \dots, \sigma_n) = \frac{1}{n!\, A^n}\, \overline{y(t)\, G_n[K_n, x(t)]} \qquad \text{where } K_n^* = u(\tau_1 - \sigma_1) \cdots u(\tau_n - \sigma_n)}$$

Note that in this calculation we have no restrictions whatever on the values ~~th~~ of the σ's.

To actually calculate the orthogonal G-functionals for the delay circuit would require many computations & many more would be needed to actually determine the resulting average (especially if n gets large).

General form for kernels using only one multi-dimensional delay:

To get away from the many calculations required by the method discussed on the last page, we will revert to the simple correlation method begun on page 38. Now we will derive the general n^{th} order kernel:

Note that we can write the orthogonal delay G-functional as:

$$G_n[K_n, x(t)] = y_n(t, \sigma_1, \ldots, \sigma_n) + F_n$$

where F_n is a functional of degree less than n. Thus:

$$y_n(t, \sigma_1, \ldots, \sigma_n) = G_n[K_n, x(t)] - F_n = x(t-\sigma_1)\cdots x(t-\sigma_n)$$

Then
$$\overline{y(t) y_n(t, \sigma_1, \ldots, \sigma_n)} = \sum_0^\infty{}_m \overline{G_m[h_m, x(t)]\left(G_n[K_n, x(t)] - F_n\right)}$$

$$= \underset{m \neq n, \text{ all}=0.}{\overline{G_n[h_n, x(t)]\, G_n[K_n, x(t)]}}_{(m=n)} + \sum_{m<n} \overline{G_m[h_m, x(t)]\, x(t-\sigma_1)\cdots x(t-\sigma_n)}$$

$$= n!\, A^n h_n(\sigma_1, \ldots, \sigma_n) + \sum_{m<n} \overline{G_m[h_m, x(t)]\, x(t-\sigma_1)\cdots x(t-\sigma_n)}$$

Now consider the first term of $G_m[h_m, x(t)]\, x(t-\sigma_1)\cdots x(t-\sigma_n)$

$$\int\cdots\int h_m(\tau_1, \ldots, \tau_m)\, \overline{x(t-\tau_1)\cdots x(t-\tau_m)\, x(t-\sigma_1)\cdots x(t-\sigma_n)}\, d\tau_1\cdots d\tau_m$$

This term is zero, as are all succeeding terms, if $m+n$ is odd.

If $m+n$ is even and $m<n$, then $m+n = 2m+2k \Rightarrow n = m+2k$. Thus there must always be a pair of terms $x(t-\sigma_i)\, x(t-\sigma_j)$ which ~~an~~ do not pair with a τ. The average of this term is

$$\overline{x(t-\sigma_i)\, x(t-\sigma_j)} = A\, u(\sigma_i - \sigma_j)$$

Thus, the average $\overline{G_m[h_m, x(t)]\, x(t-\sigma_1)\cdots x(t-\sigma_n)}$ is zero unless $\sigma_i = \sigma_j$, when it blows up. Considering all $m<n$, we get impulses whenever any two σ's are equal. Thus

$$\boxed{h_n(\sigma_1, \ldots, \sigma_n) = \frac{1}{n!\, A^n}\, \overline{y(t)\, y_n(t, \sigma_1, \ldots, \sigma_n)} \quad \text{if } \sigma_i \neq \sigma_j, \text{ all } i \neq j,\ j = 1, \ldots, n}$$

Synthesis of kernels:

Ortho-normal functions: (Chapter 18 of Lee's book)

~~The set $\{w_n(t)\}$, $n=0,1,\ldots\}$~~

The functions belonging to the set $\{w_n(t), n=0,1,2,\ldots\}$ are ortho-normal over the interval (a,b) if

$$\boxed{\int_a^b w_m(t)\, w_n(t)\, dt = \delta_{mn}}$$

If a function $f(t)$ can be represented over the interval as

$$f(t) = \sum_0^\infty A_n w_n(t) \quad , \text{ then } A_m = \int_a^b f(t)\, w_m(t)\, dt = A_m \int_a^b w_m(t)\, w_m(t)\, dt .$$

The set $\{w_n(t)\}$ is said to be complete if

$$\lim_{N\to\infty} \int_a^b \left[f(t) - \sum_0^N A_m w_m(t) \right]^2 dt = 0 \quad \text{for } f(t) \ni \int_a^b f^2(t)\, dt < \infty .$$

We will consider only complete orthonormal sets, using the time interval $0 \le t < \infty$. The completeness of these sets will not be proven as it is generally very difficult.

We will use functions that can be impulse responses of linear networks so that we can synthesize the expressions we get for the kernels.

The functions of a set are said to be linearly independent if no element can be expressed as a linear combination of the others.

Laguerre functions:

The set of functions $\{g(t) = (pT)^n e^{-pT}\}$ is linearly independent. We will now make a set of orthonormal functions from this set.

(1) Gram-Schmidt procedure:

This is just a repetitive procedure for finding the n^{th} orthonormal function by making it orthogonal to all others of order less than n.

$$l_0(t) = c_0 g_0(T) = c_0 e^{-pT}$$

$$1 = \int_0^\infty l_0^2(T)\,dT = c_0^2 \int_0^\infty e^{-pT} = \frac{c_0^2}{2p} \Rightarrow \boxed{c_0 = \sqrt{2p}}$$

so $\boxed{l_0(T) = \sqrt{2p}\, e^{-pT}}$

$$l_1(T) = c_1 g_0(T) + c_2 g_1(T)$$

$$0 = \int_0^\infty l_0(t)\, l_0(T)\,dT = \int_0^\infty \sqrt{2p}\, e^{-pT}\left[c_1 e^{-pT} + c_2 pT e^{-pT}\right]dT =$$

$$0 = \frac{c_1}{2p} + \frac{c_2 p}{(2p)^2} \Rightarrow 2c_1 + c_2 = 0$$

$$1 = \int_0^\infty l_1^2(T)\,dT = \int_0^\infty c_1^2\left[2pTe^{-pT} - e^{-pT}\right]^2 dT = c_1^2 \int_0^\infty e^{-pT}\left[2pT - 1\right]^2 dT = \frac{c_1^2}{2p}$$

So $\boxed{l_1(T) = \sqrt{2p}\,[2pT-1]\,e^{-pT}}$

$$l_2(T) = c_3 l_0(T) + c_4 l_1(T) + c_5 l_2(T)$$

We will get 3 orthogonality requirements which will give 3 equations to find the 3 constants.

This procedure is, however, the hard way. It would take quite a while to get all the functions.

So, being loyal engineers, we turn to the frequency domain:

Finding Laguerre functions in the frequency domain:

If $F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt \leftrightarrow f(t)$ and $G(\omega) \leftrightarrow g(t)$,

then Parseval's theorem states:

$$\int_{-\infty}^{\infty} f(t) g(t)\, dt = 2\pi \int_{-\infty}^{\infty} \overline{F(\omega)}\, G(\omega)\, d\omega$$

If we now Fourier transform $l_n(t)$, we get

$$L_n(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} l_n(t)\, e^{-j\omega t}\, dt$$

$$\int_{0}^{\infty} l_n(t)\, l_m(t)\, dt = 2\pi \int_{-\infty}^{\infty} \overline{L_n(\omega)}\, L_m(\omega)\, d\omega = \delta_{mn}$$

Thus, the set $\{L_n(\omega)\}$ is also orthogonal (but not normal $[2\pi]$).

Working in the frequency domain, we have

$$g_n(t) = (\rho t)^n e^{-\rho t} \leftrightarrow G_n(\omega) = \frac{1}{2\pi}\int_{0}^{\infty} (\rho t)^n e^{-(\rho + j\omega)t}\, dt$$

$$G_n(\omega) = \frac{n!\, \rho^n}{2\pi(\rho + j\omega)^{n+1}}$$

Our procedure for making the orthogonal set was to form linear combinations & make the combination orthogonal to those of lesser degree:

$$L_0(\omega) = \frac{A_0}{\rho + j\omega} \quad \text{+ normalize}$$

$$L_1(\omega) = \frac{B_0}{\rho + j\omega} + \frac{B_1}{(\rho + j\omega)^2} = \frac{A_1\, g_1(\omega)}{(\rho + j\omega)^2}$$

where the $g_1(\omega)$ will depend on orthogonality restraints:

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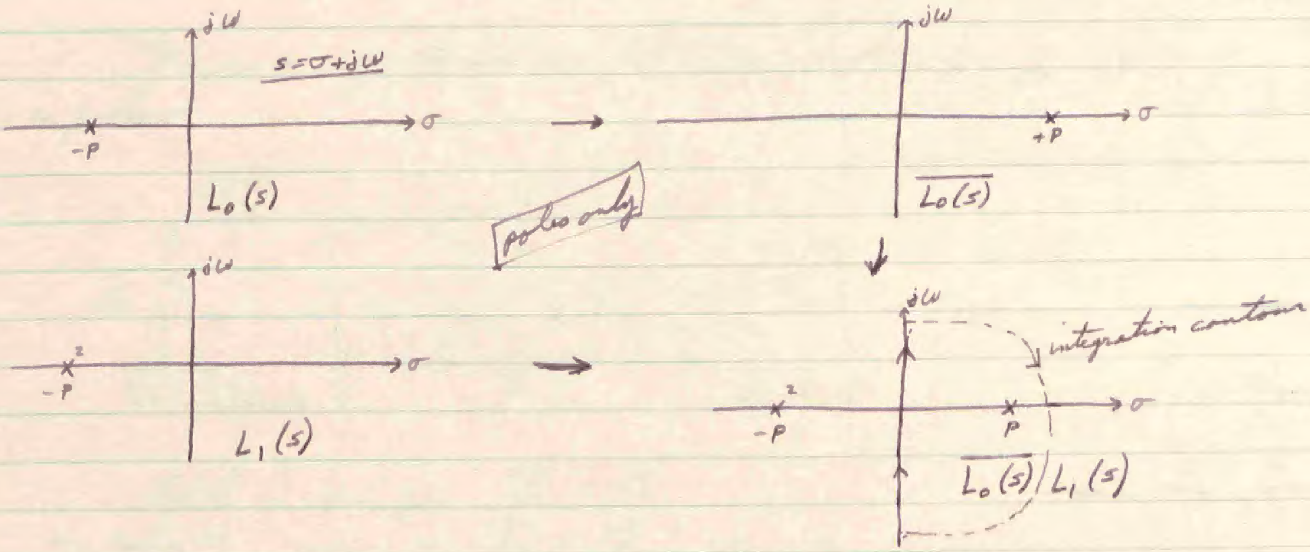
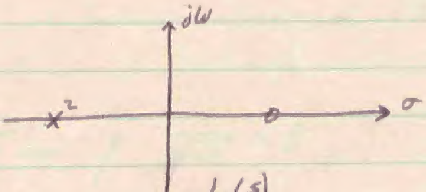
Looking at the s-plane, we see the pole plots as



For orthogonality, we must get zero when we integrate over the jω axis. If we close the contour to the right, it encloses a pole. For the integral to be zero, there cannot be a pole there. Thus $L_1(s)$ must have a zero at $s=+P$ to cancel the pole of $\overline{L_0(s)}$.



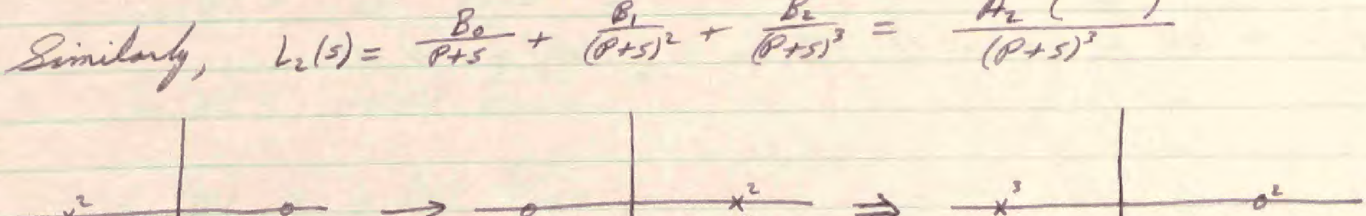
$$L_0(s) = \frac{A_0}{P+s}$$

$$\cancel{L_1(s) = B_1(P+s)}$$

$$L_1(s) = \frac{A_1(P-s)}{(P+s)^2}$$

Thus we automatically have $L_1(s)$ orthogonal to $L_0(s)$ & we need only normalize.

Similarly, $L_2(s) = \frac{B_0}{P+s} + \frac{B_1}{(P+s)^2} + \frac{B_2}{(P+s)^3} = \frac{A_2(\quad)}{(P+s)^3}$



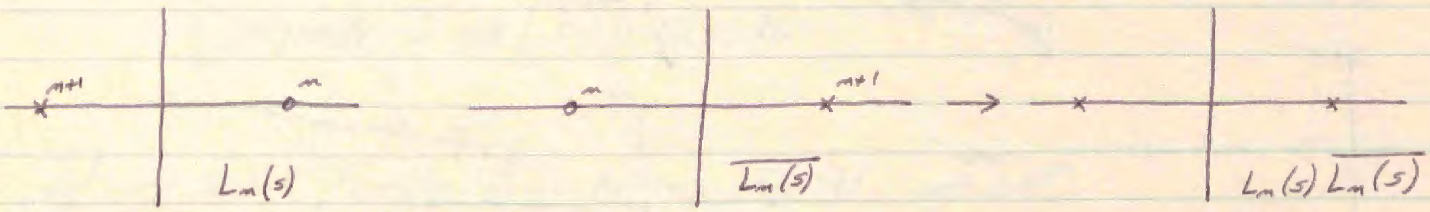
for orthogonality to $L_1(s)$

$$L_2(s) = \frac{A_2(P-s)^2}{(P+s)^3}.$$

This will also be orthogonal to $L_0(s)$ as the double zero will over-kill the single pole of $\overline{L_0(s)}$.

By inspection, we can generalize by induction to

$$\boxed{L_n(s) = \frac{A_n (P-s)^n}{(P+s)^{n+1}}}$$



To normalize, we see

$$1 = 2\pi \int_{-\infty}^{\infty} \overline{L_n(\omega)}\, L_n(\omega)\, d\omega = 2\pi \int_{-\infty}^{\infty} \frac{A_n^2 \, d\omega}{(P+j\omega)(P-j\omega)} = 2\pi A_n^2 \int_{-\infty}^{\infty} \frac{d\omega}{P^2+\omega^2} = \frac{(2\pi)^2 A_n^2}{2P}$$

$$\Rightarrow \boxed{A_n = \frac{\sqrt{2P}}{2\pi}}$$

$$\Rightarrow \boxed{L_n(s) = \frac{\sqrt{2P}}{2\pi} \frac{(P-s)^n}{(P+s)^{n+1}}}$$

$$L_n(\omega) = \frac{\sqrt{2P}}{2\pi} \frac{(P-j\omega)^n}{(P+j\omega)^{n+1}}$$

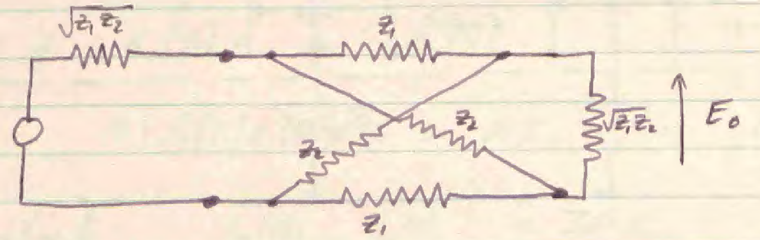
Expansion of system functions:

$$h(\tau) = \sum_0^{\infty} c_n \, l_n(\tau) \quad \leftrightarrow \quad c_n = \int_0^{\infty} h(\tau)\, l_n(\tau)\, d\tau$$

$$H(\omega) = 2\pi \sum_0^{\infty} c_n L_n(\omega) \quad \leftrightarrow \quad c_n = \int_{-\infty}^{\infty} H(\omega)\, \overline{L_n(\omega)}\, d\omega.$$

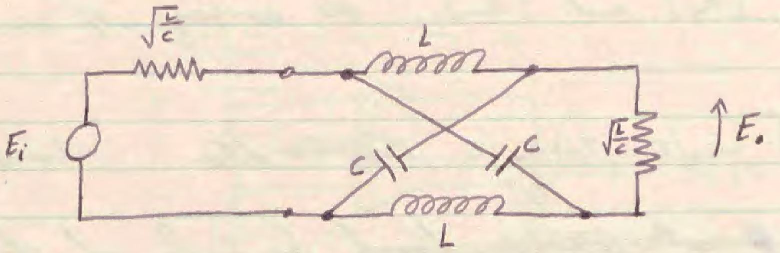
Phase shift network:

We can synthesize a network with impulse response $\frac{\rho - j\omega}{\rho + j\omega}$ as follows



$$\frac{E_0(\omega)}{E_i(\omega)} = H(\omega) = \frac{\sqrt{Z_2} - \sqrt{Z_1}}{\sqrt{Z_2} + \sqrt{Z_1}}$$

If $Z_1 = j\omega L$, $Z_2 = \frac{1}{j\omega C}$,



$$\frac{E_0}{E_i} = \frac{\frac{1}{\sqrt{LC}} - j\omega}{\frac{1}{\sqrt{LC}} + j\omega}$$

Thus, if $\frac{1}{\sqrt{LC}} = \rho$, $\quad \frac{E_0}{E_i} = H(\omega) = \frac{\rho - j\omega}{\rho + j\omega}$

Now, we note that

$$L_0(\omega) = \frac{\sqrt{2\rho}}{2\pi} \frac{1}{\rho + j\omega}$$

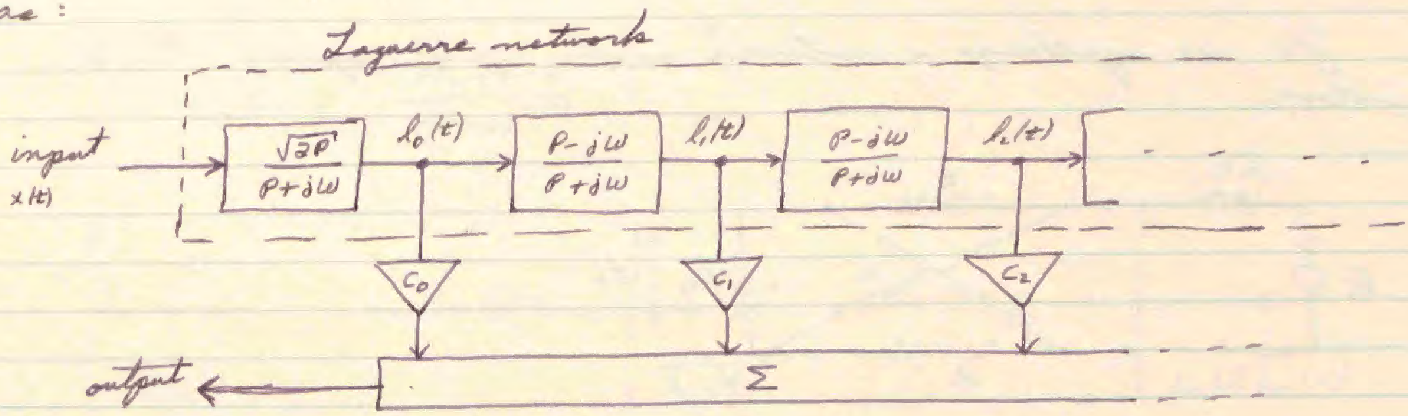
$$L_1(\omega) = L_0(\omega) \frac{\rho - j\omega}{\rho + j\omega}$$

$$L_2(\omega) = L_1(\omega) \frac{\rho - j\omega}{\rho + j\omega}$$

$\vdots$

etc.

We can then synthesize the system with impulse response $h(\tau) = \sum_{0}^{\infty} l_n(\tau)\, c_n$ as:



Synthesis of kernels:

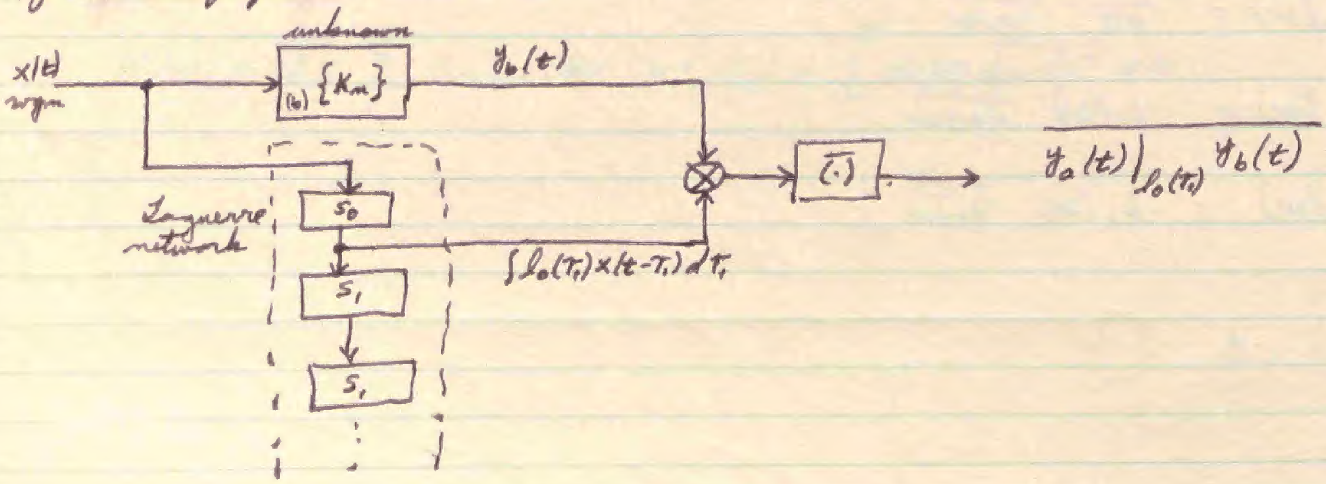
If we have two systems, $\{h_n\}$ and $\{K_n\}$, the average of the product of their output is

$$\overline{y_a(t)\, y_b(t)} = \sum_{0}^{\infty} n!\, A^n \int \cdots \int h_n(\tau_1, \dots, \tau_n)\, K_n(\tau_1, \dots, \tau_n)\, d\tau_1 \cdots d\tau_n \qquad \text{(see p. 37)}$$

If system (a) has only the ~~n-th G-functional~~ kernels of the n^{th} G-functional, then we have

$$\overline{y_a(t)\big|_{h_n}\, y_b(t)} = n!\, A^n \int \cdots \int h_n(\tau_1, \dots, \tau_n)\, K_n(\tau_1, \dots, \tau_n)\, d\tau_1 \cdots d\tau_n$$

Synthesis of first order kernel:



Now the output average is

$$\overline{y_a(t)\Big|_{l_0(T_1)} y_b(t)} = A\int l_0(T_1)\, K_1(T_1)\, dT_1$$

Now if $K_1(T_1) = \sum_{k=0}^{\infty} c_k\, l_k(T_1)$

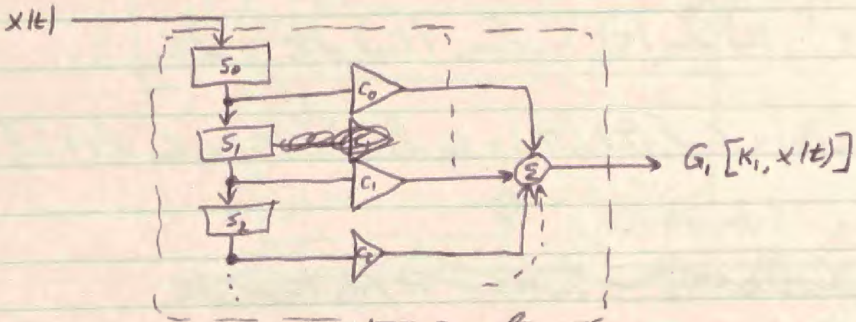
then $A\int l_0(T_1)\, K_1(T_1)\, dT_1 = A c_0 \Rightarrow \boxed{c_0 = \frac{1}{A}\, \overline{y_a(t)\Big|_{l_0(T_1)} y_b(t)}}$

Similarly, $\overline{y_a(t)\Big|_{l_n(T_1)} y_b(t)} = A c_n$

so

$$\boxed{c_n = \frac{1}{A}\, \overline{y_a(t)\Big|_{l_n(T_1)} y_b(t)}}$$

Thus we can construct the linear first order kernel:



First order kernel.

Synthesis of second order kernel:

Let $h_2(\tau_1, \tau_2) = \sum_0^\infty a_n(\tau_2)\, l_n(\tau_1)$, $\tau_1 \geq 0$

$$a_n(\tau_2) = \int_0^\infty h_2(\tau_1, \tau_2)\, l_n(\tau_1)\, d\tau_1$$

$$a_n(\tau_2) = \sum_0^\infty b_{nm}\, l_m(\tau_2) \quad , \quad \tau_2 \geq 0$$

$$\boxed{\begin{aligned} b_{nm} &= \int_0^\infty a_n(\tau_2)\, l_m(\tau_2)\, d\tau_2 = \iint_0^\infty h_2(\tau_1, \tau_2)\, l_n(\tau_1)\, l_m(\tau_2)\, d\tau_1\, d\tau_2 \\ h_2(\tau_1, \tau_2) &= \sum_0^\infty \sum_0^\infty b_{nm}\, l_n(\tau_1)\, l_m(\tau_2) \quad ; \quad \tau_1, \tau_2 \geq 0 \end{aligned}}$$

In general, we can write

$$\boxed{\begin{aligned} h_n(\tau_1, \dots, \tau_n) &= \sum_{m_1=0}^\infty \cdots \sum_{m_n=0}^\infty c_{m_1 \dots m_n}\, l_{m_1}(\tau_1) \cdots l_{m_n}(\tau_n) \\ c_{m_1 \dots m_n} &= \int_0^\infty \cdots \int_0^\infty h_n(\tau_1, \dots, \tau_n)\, l_{m_1}(\tau_1) \cdots l_{m_n}(\tau_n)\, d\tau_1 \cdots d\tau_n \end{aligned}}$$

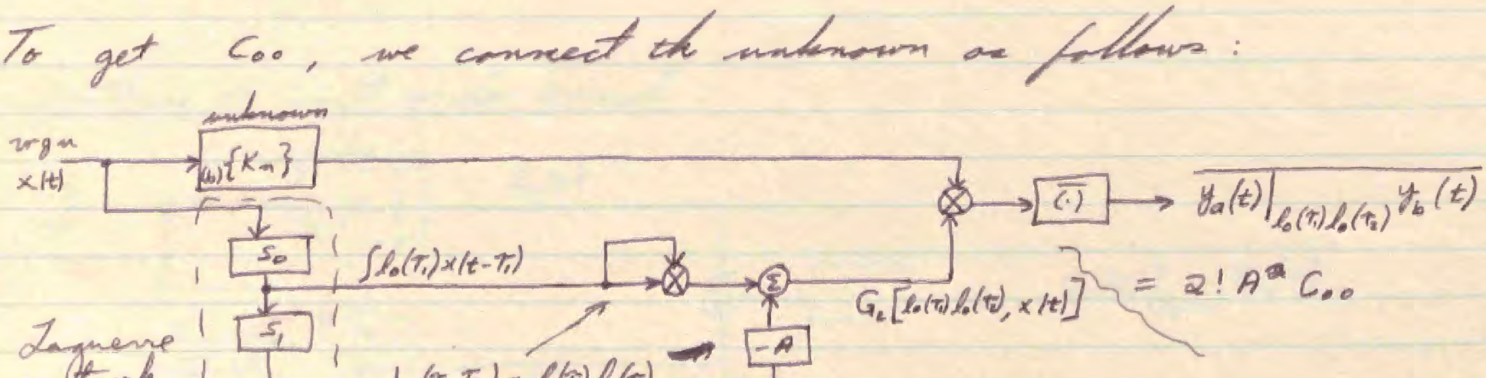
Now, if $K_2(\tau_1, \tau_2) = c_{00}\, l_0(\tau_1)\, l_0(\tau_2) + c_{11}\, l_1(\tau_1)\, l_1(\tau_2) + c_{01}\, l_0(\tau_1)\, l_1(\tau_2) + c_{10}\, l_1(\tau_1)\, l_0(\tau_2)$

~~The G fn~~

~~The 2nd G functional is~~ $\cancel{G_2[K_2, x(t)] = y_2(t) - A \int K_2(\tau_1, \tau_1)\, d\tau_1}$

$$c_{m_1 m_2} = \iint K_2(\tau_1, \tau_2)\, l_{m_1}(\tau_1)\, l_{m_2}(\tau_2)\, d\tau_1\, d\tau_2$$

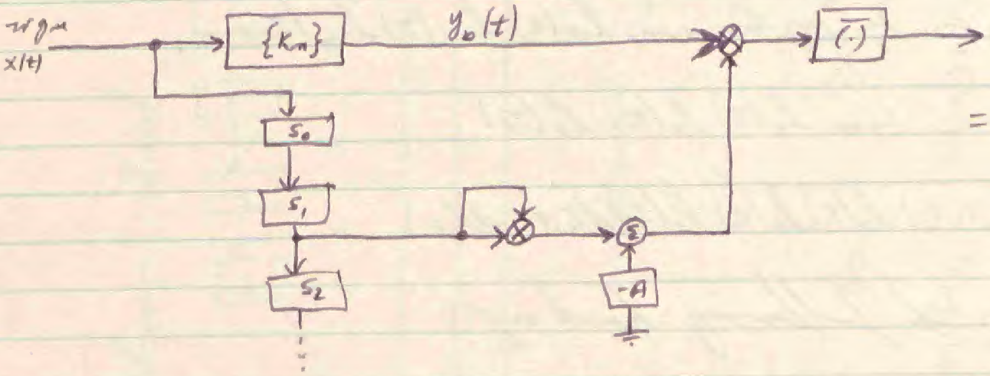
To get c_{00}, we connect the unknown as follows:



$$-A \int h_2(\tau_1, \tau_1)\, d\tau_1 = -A \int l_0(\tau_1)\, l_0(\tau_1)\, d\tau_1 = -A$$

Hence, $\boxed{c_{00} = \frac{1}{2!\, A^2}\, \overline{y_a(t)\big|_{l_0(\tau_1) l_0(\tau_2)}\, y_b(t)}}$

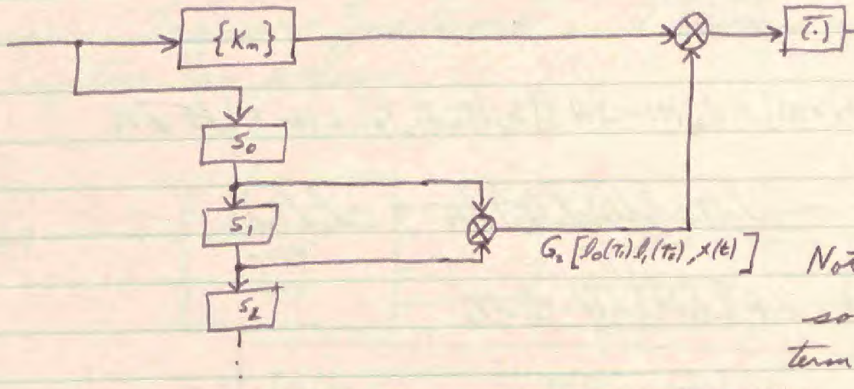
Similarly, we can compute c_{11}:



$$\overline{y_a(t)\big|_{l_1(\tau_1)\,l_1(\tau_1)}\, y_b(t)} = 2!\,A^2\, c_{11}$$

so $$\boxed{c_{11} = \frac{1}{2!\,A^2}\,\overline{y_a(t)\big|_{l_1(\tau_1)\,l_1(\tau_1)}\, y_b(t)}}$$

For computing c_{01} and c_{10}, the hook-up is a little different:



$$\overline{y_a(t)\big|_{l_0(\tau_1)\,l_1(\tau_2)}\, y_b(t)} = 2!\,A^2\, c_{01}$$

Note: $\int l_0(\tau_1)\, l_1(\tau_1)\, d\tau_1 = 0$ so we have no constant term in this second order kernel.

$$\boxed{c_{01} = c_{10} = \frac{1}{2!\,A^2}\,\overline{y_a(t)\big|_{l_0(\tau_1)\,l_1(\tau_2)}\, y_b(t)}}$$

~~[illegible]~~

Continuing on in a similar fashion, we can find all the constants c_{mn}. Then we can synthesize ~~[illegible]~~ this second order kernel with a Laguerre network & many, many amplifiers & multipliers & adders.

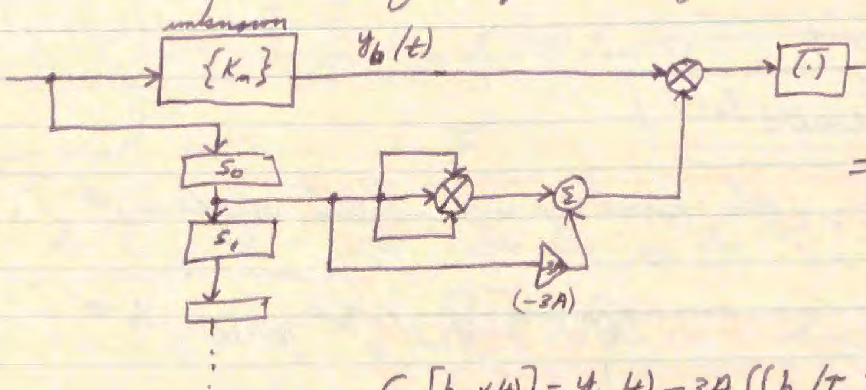
Synthesis of first term of third order kernel:

$$G_3[K_3, x(t)] \rightarrow K_3(\tau_1,\tau_2,\tau_3) = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \sum_{m_3=0}^{\infty} c_{m_1,m_2,m_3}\, l_{m_1}(\tau_1)\, l_{m_2}(\tau_2)\, l_{m_3}(\tau_3)$$

Consider the first term: $c_{000}\, l_0(\tau_1)\, l_0(\tau_2)\, l_0(\tau_3)$

$$c_{000} = \iiint K_3(\tau_1,\tau_2,\tau_3)\, l_0(\tau_1)\, l_0(\tau_2)\, l_0(\tau_3)\, d\tau_1\, d\tau_2\, d\tau_3$$

We calculate c_{000} by the following hook-up:



$$\overline{y_a(t)\Big|_{l_0(\tau_1) l_0(\tau_2) l_0(\tau_3)}\, y_b(t)} = 3!\, A^3 c_{000}$$

$$G_3[h_3, x(t)] = y_{h_3}(t) - 3A \iint h_3(\tau_1,\tau_2,\tau_2)\, x(t-\tau_1)\, d\tau_1\, d\tau_2$$

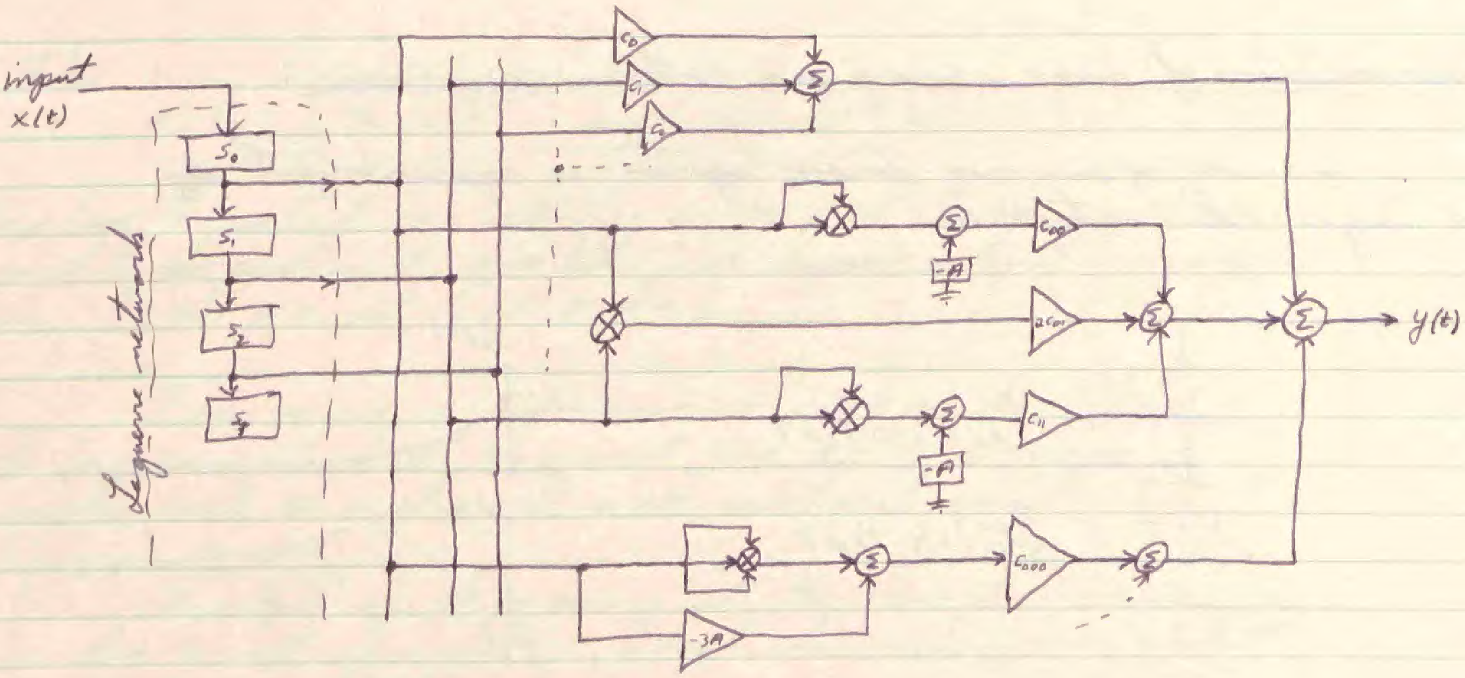
Here, $\int h_3(\tau_1,\tau_2,\tau_2)\, d\tau_2 = \int l_0(\tau_1)\, l_0(\tau_2)\, l_0(\tau_2)\, d\tau_2 = l_0(\tau_1)$

so $G_3[h_3, x(t)] = y_{h_3}(t) - 3A \int l_0(\tau_1)\, x(t-\tau_1)\, d\tau_1$

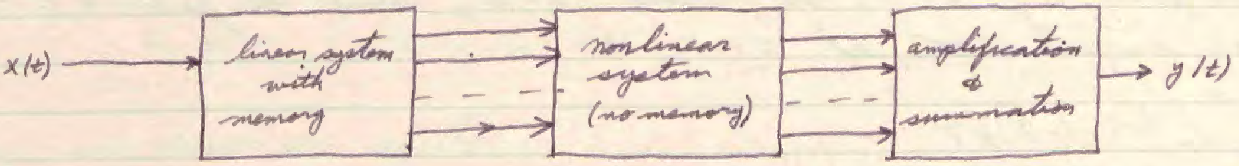
So,
$$\boxed{c_{000} = \frac{1}{3!\, A^3}\, \overline{y_a(t)\Big|_{l_0(\tau_1) l_0(\tau_2) l_0(\tau_3)}\, y_b(t)}}$$

In principle, at least, if not practically, we could get the complete expansion of K_3, K_4, ... etc. until we had each kernel fully synthesized by this technique.

At the top of the next page is a *partial* picture of what our final system would look like:



Note that the system has memory only in the linear "Laguerre network". Thus we can ~~dege~~ represent the system as follows:



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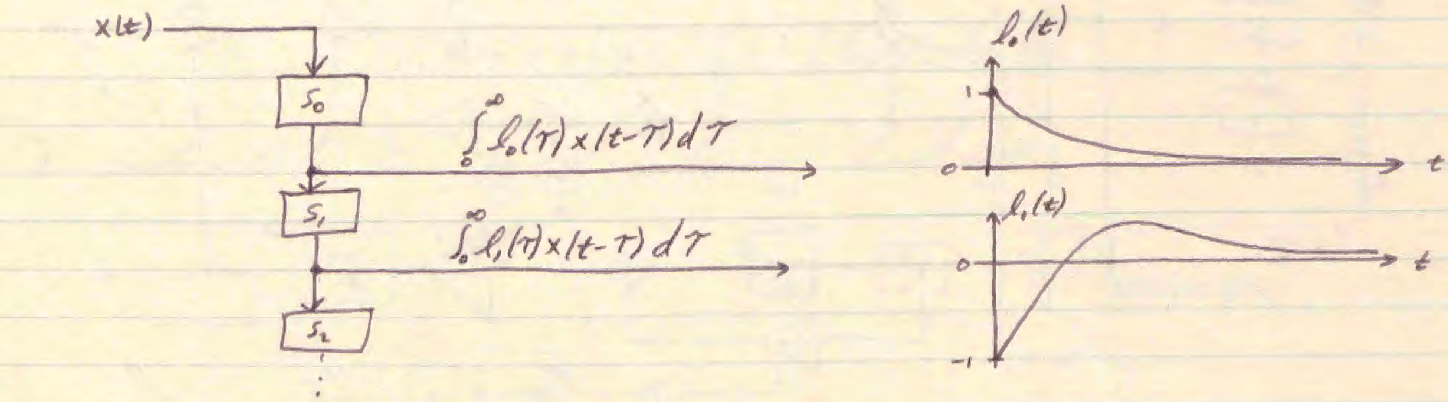
Interpretation of linear memory and Volterra expansion:

The memory of our non-linear system is represented by a Laguerre network:



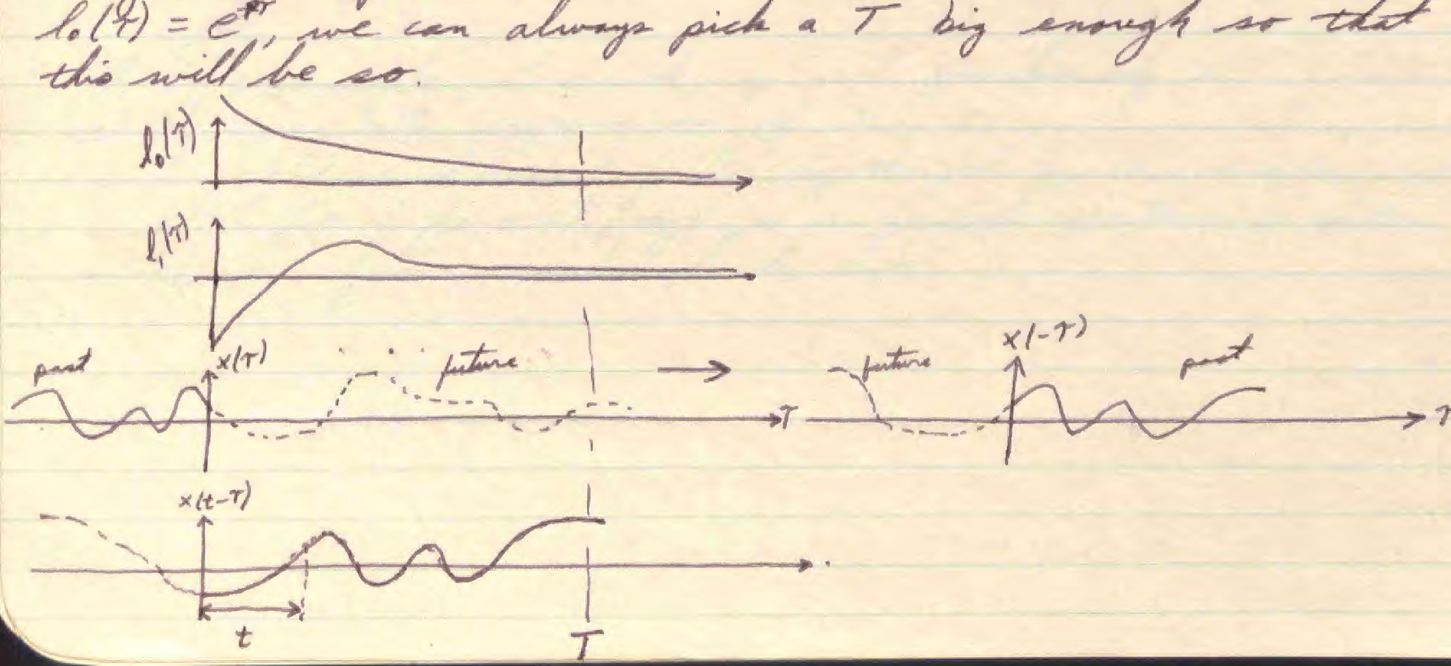
The output of the n^{th} section of the Laguerre network is

$$\int_0^\infty l_n(\tau)x(t-\tau)d\tau$$

Let the past be $t<0$, the present $t=0$, and the future $t>0$. Then the output now, at $t=0$, is

$$\int_0^\infty l_n(\tau)x(-\tau)d\tau$$

Now, suppose that the output of the linear box depends importantly only on the input back T seconds, $T<\infty$. Note that since $l_0(\tau)=e^{-\tau}$, we can always pick a T big enough so that this will be so.



Then,

$$x(-\tau) = \sum_{0}^{\infty} v_n l_n(\tau) \qquad \text{where } v_n = \int_0^T x(-\tau) l_n(\tau) d\tau$$

We have to restrict our attention on the past to the region $0 < \tau < T$ so that the ~~past~~ input will ~~be~~ be finite ~~square~~ integrable square. [i.e., $\int_0^T x^2(-\tau)d\tau < \infty$ but $\int_0^\infty x^2(-\tau)d\tau \to \infty$].

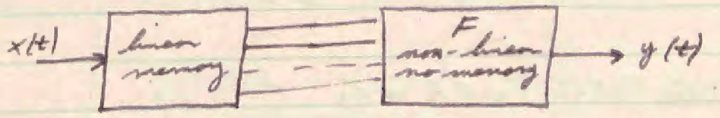
Thus, as we consider more & more of the outputs of the Laguerre networks at the present time $t=0$, we can approximate the past input with arbitrarily small mean square error.

The output of the memory at the present time is then a set of numbers which characterize the past input to the system. As time goes on, these numbers will change since the past input has been modified. Thus, we can write

$$x(t-\tau) = \sum_{0}^{\infty} v_n(t) l_n(\tau)$$

$$v_n(t) = \int_0^T x(t-\tau) l_n(\tau) d\tau$$

We can now draw our system



$y(t) = F[v_0(t), v_1(t), \ldots]$, where $F[\cdot]$ is an instantaneous non-linear function.

If we expand $F[v_0(t), \ldots]$ in a power series, we get

$$y(t) = \sum_{0}^{\infty} C_{m_1} v_{m_1}(t) + \sum_{0}^{\infty}\sum_{0}^{\infty} C_{m_1 m_2} v_{m_1}(t) v_{m_2}(t) + \cdots$$

Looking at the first term,

$$\sum_{m_1=0}^{\infty} c_{m_1} v_{m_1}(t) = c_0 \int_0^{\infty} l_0(\tau) x(t-\tau)\, d\tau + c_1 \int_0^{\infty} l_1(\tau) x(t-\tau)\, d\tau + \cdots$$

$$= \sum_{m_1=0}^{\infty} c_{m_1} \int_0^{\infty} l_{m_1}(\tau) x(t-\tau)\, d\tau$$

$$= \int_0^{\infty} \sum_{0}^{\infty} c_{m_1} l_{m_1}(\tau)\, x(t-\tau)\, d\tau$$

$$= \int_0^{\infty} h_1(\tau) x(t-\tau)\, d\tau$$

where $h_1(\tau) = \sum_{0}^{\infty} c_{m_1} l_{m_1}(\tau)$

The second term is

$$\sum_{0}^{\infty} \sum_{0}^{\infty} c_{m_1,m_2} v_{m_1}(t) v_{m_2}(t) = \sum\sum c_{m_1,m_2} \int_0^{\infty} l_{m_1}(\tau_1) x(t-\tau_1)\, d\tau_1 \int_0^{\infty} l_{m_2}(\tau_2) x(t-\tau_2)\, d\tau_2$$

$$= \int_0^{\infty}\int_0^{\infty} \sum\sum c_{m_1,m_2} l_{m_1}(\tau_1) l_{m_2}(\tau_2)\, x(t-\tau_1) x(t-\tau_2)\, d\tau_1\, d\tau_2$$

$$= \int_0^{\infty}\int_0^{\infty} h_2(\tau_1,\tau_2) x(t-\tau_1) x(t-\tau_2)\, d\tau_1\, d\tau_2$$

where $h_2(\tau_1,\tau_2) = \sum_{0}^{\infty} \sum_{0}^{\infty} c_{m_1,m_2} l_{m_1}(\tau_1) l_{m_2}(\tau_2)$

Thus

$$y(t) = \int h_1(\tau_1) x(t-\tau_1)\, d\tau_1 + \iint h_2(\tau_1,\tau_2) x(t-\tau_1) x(t-\tau_2)\, d\tau_1\, d\tau_2 + \cdots$$

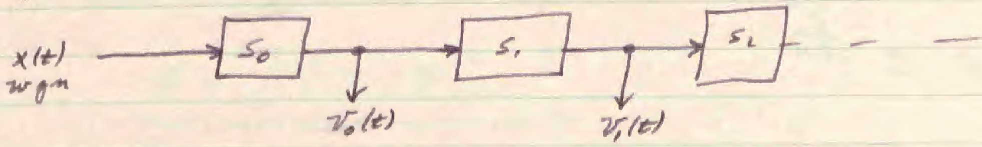
$$= \sum_{0}^{\infty} \int \cdots \int h_n(\tau_1,\ldots,\tau_n) x(t-\tau_1) \cdots x(t-\tau_n)\, d\tau_1 \cdots d\tau_n$$

Thus the Volterra representation a power series expansion of the output of the instantaneous nonlinear element of the system in terms of the inputs to that box from the linear memory.

Some properties of the linear memory outputs:

For the sake of practicality, we would generally terminate the memory network chain at N orthogonal terms. For a given system, and input, we can generally find a set of orthogonal functions that gives an optimal representation of the input for a finite number of functions from that set.
It is the non-linear element that changes from system to system, not the memory.

We will stay with the Laguerre network, although in general, another set of orthonormal functions might be better.



$$\boxed{\overline{v_n(t)} = \int_0^\infty l_n(\tau)\overline{\cancel{x_n}(t-\tau)}\,d\tau = 0}$$

$$\overline{v_m(t)v_n^*(t)} = \iint l_m(\tau_1)\,l_n(\tau_2)\,\overline{x(t-\tau_1)x(t-\tau_2)}\,d\tau_1 d\tau_2 = A\iint l_m(\tau_1)\,l_n(\tau_2)\,u(\tau_1-\tau_2)\,d\tau_1 d\tau_2$$

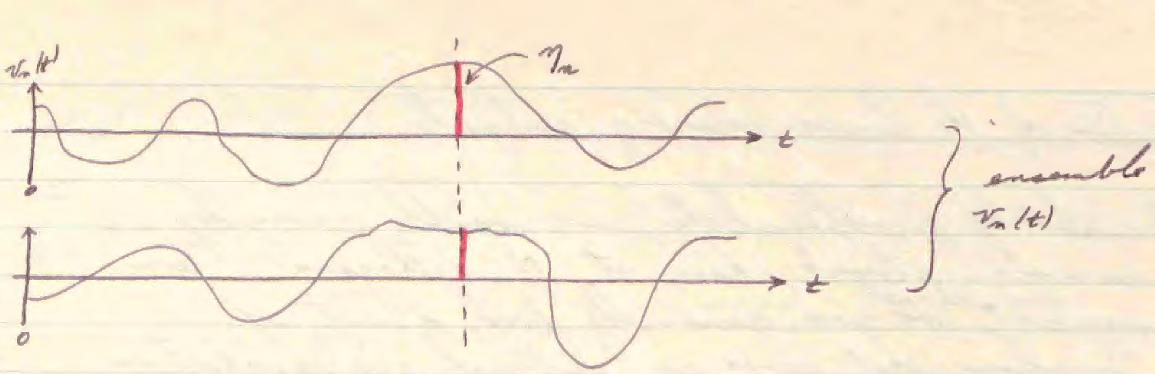
$$= A\int l_m(\tau_1)\,l_n(\tau_1)\,d\tau_1$$

so
$$\boxed{\begin{aligned}\overline{v_m(t)v_n(t)} &= 0, \quad m \neq n \\ \overline{v_n^2(t)} &= A\end{aligned}}$$

$$\overline{v_n^K(t)} = \int\cdots\int l_n(\tau_1)\cdots l_n(\tau_K)\,\overline{x(t-\tau_1)\cdots x(t-\tau_K)}\,d\tau_1\cdots d\tau_K$$ see page 24

$$= \begin{cases} A^{\frac{K}{2}}\left[(K-1)(K-3)\cdots 1\right]\int\cdots\int l_n(\tau_1)\,l_n(\tau_1)\cdots l_n(\tau_{\frac{K}{2}})\,l_n(\tau_{K/2})\,d\tau_1\cdots d\tau_{K/2}, & \underline{K \text{ even}} \\ 0, & \underline{K \text{ odd}} \end{cases}$$

$$\boxed{\overline{v_n^K(t)} = \begin{cases} A^{K/2}(K-1)(K-3)\cdots 1, & K \text{ even} \\ 0, & K \text{ odd} \end{cases}}$$



$$\overline{\eta_n^k} = \overline{v_n^k(t)} \Rightarrow \overline{\eta_n} = 0 \;,\quad \sigma_{\eta_n}^2 = A$$

Now since we know η_n is a gaussian random variable, the first two moments completely specify the distribution:

$$P_{\eta_n}(y_n) = \frac{1}{\sqrt{2\pi A}}\, e^{-\frac{y_n^2}{2A}}$$

Also, $\quad \overline{v_m(t)\,v_n(t)} = \overline{\eta_m \eta_n} = 0$

For gaussian random variables this is a sufficient condition that the two variables be statistically independent. Thus we note that <u>the outputs $v_n(t)$ are gaussian & are statistically independent of each other.</u>

<u>Proofs that $\overline{\eta_m \eta_n} = 0 \Rightarrow$ statistical independence for gauss r.v.</u>:

$$M_\eta(\alpha_1,\dots,\alpha_N) = e^{-\frac{1}{2}\sum_1^N \sum_1^N \overline{\eta_n \eta_m}\,\alpha_n \alpha_m}$$

(see page 21)

<u>Now if $\overline{\eta_n \eta_m} = 0$</u>, $\quad M_\eta(\alpha_1,\dots,\alpha_N) = e^{-\frac{1}{2}\sum_1^N \overline{\eta_m^2}\,\alpha_m^2} = e^{-\frac{1}{2}\sum_1^N A\alpha_m^2} = \prod_{1=m}^{N} e^{-\frac{1}{2}A\alpha_m^2}$

Thus $\quad P_\eta(y_1,\dots,y_n) = \frac{1}{(2\pi)^N}\int\cdots\int M_\eta(\alpha_1,\dots,\alpha_N)\, e^{-j\alpha_1 y_1 - \cdots - j\alpha_N y_N}\, d\alpha_1 \dots d\alpha_N$

$$= \frac{1}{(2\pi)^N}\prod_{n=1}^{N}\int e^{-\frac{1}{2}A\alpha_m^2 - j\alpha_m y_m}\, d\alpha_m$$

$$= P_{\eta_1}(y_1)\cdots P_{\eta_N}(y_N)$$

$\Rightarrow$ η_m & η_n are statistically independent.

Study of nonlinear element on ensemble basis:

~~We know we~~

Let η be the random variable that is the input to the system at a particular time t. Let ξ be the output random variable at that time.

We know that we can write

$$y(t) = F[v_0(t), v_1(t), \dots]$$

where the nonlinear function F is an instantaneous operation. Thus on an ensemble basis, we can write

$$\underline{\xi = F[\eta_0, \eta_1, \dots]}.$$

We will now try to develop a multi-dimensional power series expansion of ξ in terms of the memory outputs $\eta_0, \eta_1, \dots$

We will introduce Hermite polynomials & Hermite functions to get at this expansion. Hence a brief side-track:

Hermite polynomials & Hermite functions:

References:

The Fourier Integral and Certain of its Applications, Norbert Wiener, pp 51-67.

Mathematical Methods of Statistics, H. Cramér, p 133.

Let ~~$\psi_n(x) = x^n e^{-\frac{x^2}{4A}}$, $n = 0, 1, \dots$~~

~~We will~~

We want to ~~ex~~ develop from the set $\{\psi_n(x)\} = \{x^n e^{-\frac{x^2}{4A}}\}$ a set $\{\phi_n(x)\}$ of functions which are orthonormal over the range $(-\infty, \infty)$. This treatment is different than the standard treatments which treat only the case $A=1$. The x's will refer to the variables $\eta_0, \eta_1, \ldots$ in our model so we require the doubly infinite interval.

We can procede to develop the set by requiring

$$\int_{-\infty}^{\infty} \phi_n(x)\,\phi_m(x)\,dx = \delta_{mn}$$

Two helpful integrals are:

$$\frac{1}{\sqrt{2\pi A}}\int_{-\infty}^{\infty} x^n e^{-\frac{x^2}{2A}}\,dx = A^{n/2}(n-1)(n-3)\cdots 1$$

or $$\boxed{\int_{-\infty}^{\infty} x^n e^{-\frac{x^2}{2A}}\,dx = \sqrt{2\pi}\,A^{\frac{n+1}{2}}(n-1)(n-3)\cdots 1}$$

$$\boxed{\int_{-\infty}^{\infty} e^{-\frac{x^2}{2A}}\,dx = \sqrt{2\pi A}}$$

$$1 = a_0^2\int_{-\infty}^{\infty}\phi_0^2(x)\,dx = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2A}}\,dx = \sqrt{2\pi A} \Rightarrow \underline{a_0 = \frac{1}{(2\pi A)^{1/4}}}$$

$$\underline{\phi_0(x) = \frac{1}{(2\pi A)^{1/4}} e^{-\frac{x^2}{4A}}}$$

Let $\phi_1(x) = (a_1 + b_1 x)e^{-\frac{x^2}{4A}}$

$$0 = a_0\int_{-\infty}^{\infty}(a_1 + b_1 x)e^{-\frac{x^2}{2A}}\,dx = a_0 a_1\sqrt{2\pi A} \Rightarrow \underline{a_1 = 0}$$

$$1 = b_1^2\int_{-\infty}^{\infty} x^2 e^{-\frac{x^2}{2A}}\,dx = b_1^2\sqrt{2\pi}\,A^{3/2} \Rightarrow b_1 = \frac{1}{(2\pi A)^{1/4}\sqrt{A}}$$

or

$$\underline{\phi_1(x) = \frac{1}{(2\pi A)^{1/4}\sqrt{A}}\, x e^{-\frac{x^2}{4A}}}$$

$$\Phi_2(x) = (a_2 + b_2 x + c_2 x^2)\, e^{-\frac{x^2}{4A}}$$

(1) $0 = \int \Phi_2(x)\Phi_0(x)\,dx \Rightarrow a_2 = -c_2 A$

(2) $0 = \int \Phi_2(x)\Phi_1(x)\,dx \Rightarrow b_2 = 0$

(3) $1 = \int \Phi_2^2(x)\,dx \Rightarrow c_2 = \frac{1}{(2\pi A)^{1/4}\sqrt{2}\,A}$

$$\Rightarrow \Phi_2(x) = \frac{1}{(2\pi A)^{1/4}\sqrt{2}\,A}(-A + x^2)\, e^{-\frac{x^2}{4A}}$$

But changing the sign of $\Phi_2(x)$ will not alter relations (1), (2), or (3), so we let

$$\underline{\Phi_2(x) = \frac{1}{(2\pi A)^{1/4}\sqrt{2}\,A}(x^2 - A)\, e^{-\frac{x^2}{4A}}}$$

It can be shown in general that

$$\boxed{\Phi_n(x) = \frac{1}{(2\pi A)^{1/4}\sqrt{n!}\,A^{n/2}} H_n(x)\, e^{-\frac{x^2}{4A}} \quad ; n = 0, 1, \ldots}$$

<u>$\Phi_n(x)$ is a Hermite function.</u>

<u>$H_n(x)$ is a Hermite polynomial.</u>

$H_0(x) = 1$
$H_1(x) = x$
$H_2(x) = x^2 - A$
$H_3(x) = x^3 - 3Ax$
$H_4(x) = x^4 - 6Ax^2 + 3A^2$
$H_5(x) = x^5 - 10Ax^3 + 15A^2x$
$H_6(x) = x^6 - 15Ax^4 + 45A^2x^2 - 15A^3$

$$\boxed{H_n(x) = A^n e^{\frac{x^2}{2A}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2A}}}$$

We can get these results from a standard table of Hermite polynomials by putting in the A in the proper powers [$A = \sigma^2$, exponent x plus exponent σ = order of polynomial].

A recursive formula for $H_n(x)$ can be found by noting that

$$A^n \frac{d^n}{dx^n} e^{-\frac{x^2}{2A}} = (-1)^n H_n(x) e^{-\frac{x^2}{2A}}$$

$$\begin{aligned} A^{n+1} \frac{d^{n+1}}{dx^{n+1}} e^{-\frac{x^2}{2A}} &= (-1)^{n+1} H_{n+1}(x) e^{-\frac{x^2}{2A}} \\ &= A \frac{d}{dx}\left[(-1)^n H_n(x) e^{-\frac{x^2}{2A}}\right] \\ &= A(-1)^n \left[H'(x) - \frac{x}{A} H_n(x)\right] e^{-\frac{x^2}{2A}} \\ &= (-1)^n \left[A H'(x) - x H_n(x)\right] e^{-\frac{x^2}{2A}} \end{aligned}$$

$$\Rightarrow \boxed{H_{n+1}(x) = x H_n(x) - A H_n'(x)}$$

<u>Series expansion for ξ</u>:

From page 61 we recall that we can write

$$\xi = F[\eta_0, \eta_1, \dots].$$

We would like to expand ξ in a series of the form

$$\xi = \sum_0^\infty a_i \eta_i + \sum_0^\infty \sum_0^\infty a_{ij} \eta_i \eta_j + \sum_0^\infty \sum_0^\infty \sum_0^\infty a_{ijk} \eta_i \eta_j \eta_k + \dots$$

Let $\underline{\eta}$ be the multi-dimensional vector

$$\underline{\eta} = (\eta_0, \eta_1, \dots)$$

We now would like to develop a set of functions of the variables $\eta_0, \eta_1, \dots$, such that the functions are orthogonal. Then we can expand ξ in an orthogonal series.

We will call this set

$$\{ Q^{(n)}_{ijk\ldots}(\underline{\eta}) \}$$

where $\overline{Q^{(n)}_{ijk\ldots}(\underline{\eta})\, Q^{(m)}_{ijk\ldots}(\underline{\eta})} = 0, \quad m \neq n.$

We will now procede to develop a set of polynomials in $\eta_0, \eta_1, \ldots$, such that the function of n^{th} order is orthogonal to all polynomials in $\eta_0, \eta_1, \ldots$ of degree less than n:

Let $Q^{(0)}(\underline{\eta}) = 1$, somewhat arbitrarily.

Let $Q^{(1)}_i(\underline{\eta}) = c_i + a_i \eta_i$

To find the constraints on these polynomials due to orthogonality requirements, we must compute the averages on an ensemble basis. The η's are random variables, not the output of a random process. We previously found

$$P_{\eta_i}(y_i) = \frac{1}{\sqrt{2\pi A}} e^{-\frac{y_i^2}{2A}}$$

and the η's are statistically independent.

Now, requiring $Q^{(1)}_i(\underline{\eta})$ to be orthogonal to all constants gives

$$0 = \overline{Q^{(1)}_i(\underline{\eta})\, C} = \overline{(c_i + a_i \eta_i)\, C}$$

$$0 = C \int_{-\infty}^{\infty} (c_i + a_i y_i) \frac{1}{\sqrt{2\pi A}} e^{-\frac{y_i^2}{2A}} dy_i = C c_i \Rightarrow \underline{c_i = 0}$$

Thus

$$\boxed{Q^{(1)}_i(\underline{\eta}) = a_i \eta_i}$$

Let $Q^{(2)}_{ij}(\underline{\eta}) = c_{ij} + a_i \eta_i + a_j \eta_j + a_{ij} \eta_i \eta_j$

① $\overline{(c_{ij} + a_i \eta_i + a_j \eta_j + a_{ij} \eta_i \eta_j) C} = 0$

$$0 = C \iint (c_{ij} + a_i \eta_i + a_j \eta_j + a_{ij} \eta_i \eta_j) P_{\eta_i \eta_j}(y_i, y_j)\, dy_i\, dy_j$$

$$P_{\eta_i \eta_j}(y_i, y_j) = \frac{1}{\sqrt{2\pi A}} e^{-\frac{y_i^2}{2}} \frac{1}{\sqrt{2\pi A}} e^{-\frac{y_j^2}{2A}} = P_{\eta_i}(y_i) P_{\eta_j}(y_j)$$

so $0 = C[c_{ij} + a_{ij} A \delta_{ij}] \Rightarrow \underline{c_{ij} = -A a_{ij} \delta_{ij}}$

② $\overline{(c_{ij} + a_i \eta_i + a_j \eta_j + a_{ij} \eta_i \eta_j) \eta_k} = 0$

Now η_k is just another of the gaussian random variables & has the same distribution as η_i and η_j. So:

$$0 = a_i \overline{\eta_i \eta_k} + a_j \overline{\eta_j \eta_k} = a_i A \delta_{ik} + a_j A \delta_{jk} \Rightarrow \underline{0 = a_i \delta_{ik} + a_j \delta_{jk}}.$$

Now if $i \neq k \neq j$, this expression is identically zero. But it must also be true if $i = j$:

$a_i \delta_{ik} + a_i \delta_{ik} = 0$, all $k \Rightarrow \underline{a_i = 0}$

Similarly, $\underline{a_j = 0}$.

and $$\boxed{Q^{(2)}_{ij}(\underline{\eta}) = a_{ij}(\eta_i \eta_j - A \delta_{ij})}$$

Now, assume (we will prove this later) that if n is even, $Q^{(n)}_{ijk\ldots}(\underline{\eta})$ has only terms involving an even number of products of η's. Similarly, if n is odd, $Q^{(n)}_{ijk\ldots}(\underline{\eta})$ has only odd terms.

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(3) η_k has same type Gauss dist as η_i, η_j etc.

$\overline{c_{ij}\eta_k} = 0$

$\overline{a_i \eta_i \eta_k} = a_i A \delta_{ik}$

$\overline{a_j \eta_j \eta_k} = a_j A \delta_{jk}$

$\Rightarrow a_i A \delta_{ik} + a_j A \delta_{jk} = 0$

$\Rightarrow a_i \delta_{ik} + a_j \delta_{jk} = 0$

$\overline{a_{ij}\eta_i \eta_j \eta_k} = 0$

if $i=j$, $a_i \delta_{ik} + a_i \delta_{ik} = 0 \Rightarrow a_i = 0$ + $a_j = 0$

so $Q_{ij}^{(2)}(\underline{\eta}) = a_{ij}\eta_i\eta_j - A a_{ij}\delta_{ij} = a_{ij}(\eta_i\eta_j - A\delta_{ij})$

↑ Hermite poly in 2d

$Q_{ijk}^{(3)}(\underline{\eta}) = c_{ijk} + a_i\eta_i + a_j\eta_j + a_k\eta_k + \dots$

Assume (will prove later) even degree $Q^{(n)}$ will have only ~~odd~~ even terms

$Q_{ijk\ldots}^{(n)}(\underline{\eta})$ has only $\left\{\begin{matrix}\text{even terms if } n \text{ is even} \\ \text{odd} \quad \text{"} \quad \text{"} \quad \text{odd}\end{matrix}\right\}$

so let $Q_{ijk}^{(3)}(\underline{\eta}) = b_{ijk}\eta_i + b_{jki}\eta_j + b_{kij}\eta_k + a_{ijk}\eta_i\eta_j\eta_k$

obv orthog all consts ~~all [illegible]~~ ↑ symm (mistake)

orthog to first degree polys

$b_{ijk}\eta_i + b_{jik}\eta_j + b_{kij}\eta_k$

$\overline{(b_{ijk}\eta_i + b_{jki}\eta_j + b_{kij}\eta_k + a_{ijk}\eta_i\eta_j\eta_k)\eta_l} = 0$

$0 = b_{ijk} A\delta_{il} + b_{jki} A\delta_{jl} + b_{kij} A\delta_{kl} + a_{ijk}\overline{\eta_i\eta_j\eta_k\eta_l}$

$\overline{\eta_i\eta_j\eta_k\eta_l} = A^2\left[\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right]$

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if all subscripts diff, identically zero.

regroup : $0 = (A b_{ijk} + A^2 a_{ijk}\delta_{jk})\delta_{il}$
$+ (A b_{jki} + A^2 a_{ijk}\delta_{ik})\delta_{jl}$
$+ (A b_{kij} + A^2 a_{ijk}\delta_{ij})\delta_{kl}$

l can be i, j, k or diff
zero for all conds on equality of i, j, k, l
$\Rightarrow$ each bracket $= 0$
$\Rightarrow$ $b_{ijk} = -A\, a_{ijk}\,\delta_{jk}$
$b_{jki} = -A\, a_{ijk}\,\delta_{ik}$
$b_{kij} = -A\, a_{ijk}\,\delta_{ij}$

so $Q^{(3)}_{ijk}(\underline{\eta}) = a_{ijk}\left[\eta_i\eta_j\eta_k - A(\eta_i\delta_{jk} + \eta_j\delta_{ik} + \eta_k\delta_{ij})\right]$

$i = j = k$ $\Leftrightarrow$ Hermite poly.

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$$y(t) = F[v_0(t), v_1(t), \dots] = \sum_{0}^{\infty} G_n[h_n, x(t)]$$

$$\xi = F(\eta_0, \eta_1, \dots] = \sum_{0}^{\infty} Q^{(n)}_{ijk\dots}(\underline{\eta})$$

How are Q's related to Hermite polys?

Proced for develop Q_i

$Q_i^{(1)}(\eta) = c_i + a_i \eta_i$

$Q_i^{(1)}(\eta)$ orthog all consts

$\overline{Q_i^{(1)}(\eta)\, c} = 0$ [ensemb calc]

$c_i = 0$ from $\int_{-\infty}^{\infty} (c_i + a_i\eta_i) e^{-\frac{\eta_i^2}{2A}} d\eta_i = 0$ (1)

Normalize $Q_i^{(1)}(\eta)$

$a_i^2 \int_{-\infty}^{\infty} \eta_i^2 \frac{1}{\sqrt{2\pi A}} e^{-\frac{\eta_i^2}{2A}} d\eta_i = 1$ (2)

$\boxed{a_i^2 = \frac{1}{A}}$

Let $Q_i^{(1)*}(\eta)$ be $Q_i^{(1)}(\eta)$ norm

$\boxed{Q_i^{(1)*}(\eta) = \frac{1}{\sqrt{A}}[\eta_i]}$

$\boxed{b_1^2 = \frac{a_i^2}{\sqrt{2\pi A}}}$

orthonormalize $\{x^n e^{-\frac{x^2}{4A}}\}$ n = 0, 1, ...

$\phi_1(x) = (a_1 + b_1 x) e^{-\frac{x^2}{4A}}$ $\phi_0(x) = \frac{1}{(2\pi A)^{1/4}} e^{-\frac{x^2}{4A}}$

$\phi_1(x)$ orthog

$\frac{1}{(2\pi A)^{1/4}} \int_{-\infty}^{\infty} (a_1 + b_1 x) e^{-\frac{x^2}{4A}} e^{-\frac{x^2}{4A}} dx = 0$

$\frac{1}{(2\pi A)^{1/4}} \int_{-\infty}^{\infty} (a_1 + b_1 x) e^{-\frac{x^2}{2A}} dx = 0$ (1') $\Rightarrow a_1 = 0$

Normalize $\phi_1(x)$

$b_1^2 \int x^2 e^{-\frac{x^2}{2A}} dx = 1$ (2')

$\boxed{b_1^2 = \frac{1}{(2\pi A)^{1/2}} \frac{1}{A}}$

$\phi_1(x) = \frac{1}{\sqrt[4]{2\pi A}} \frac{1}{\sqrt{A}} [x] e^{-\frac{x^2}{4A}}$

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Now look at 2nd order

$$Q_{ij}^{(2)}(\underline{\eta}) = c_{ij} + a_i \eta_i + a_j \eta_j + a_{ij} \eta_i \eta_j$$

$$Q_{ii}^{(2)}(\underline{\eta}) = c_{ii} + 2a_i \eta_i + a_{ii} \eta_i^2$$

$Q_{ii}^{(2)}(\underline{\eta})$ orthog to const C

$$\overline{(c_{ii} + 2a_i \eta_i + a_{ii} \eta_i^2) C} = 0$$

$$0 = \int_{-\infty}^{\infty} [c_{ii} + 2a_i y_i + a_{ii} y_i^2] \frac{1}{\sqrt{2\pi A}} e^{-\frac{y_i^2}{2A}} dy_i \quad (3)$$

$$0 = \overline{Q_{ii}^{(2)}(\underline{\eta})\, \eta_i}$$

$$0 = \int_{-\infty}^{\infty} [c_{ii} + 2a_i y_i + a_{ii} y_i^2] y_i \frac{1}{\sqrt{2\pi A}} e^{-\frac{y_i^2}{2A}} dy_i \quad (4)$$

Normalized $Q_{ii}^{(2)}(\underline{\eta})$

$$a_{ii}^2 \int_{-\infty}^{\infty} (y_i^2 - A)^2 \frac{1}{\sqrt{2\pi A}} e^{-\frac{y_i^2}{2A}} dy_i = 1 \quad (5)$$

$$a_{ii} = \frac{1}{A}\frac{1}{\sqrt{2}}$$

$$\Phi_2(x) = (a_2 + b_2 x + c_2 x^2) e^{-\frac{x^2}{4A}}$$

$$\int_{-\infty}^{\infty} \Phi_2(x) \Phi_0(x) dx = 0$$

$$a_0 \int_{-\infty}^{\infty} (a_2 + b_2 x + c_2 x^2) e^{-\frac{x^2}{2A}} dx = 0 \quad (3')$$

$$\int_{-\infty}^{\infty} \Phi_1(x) \Phi_2(x) dx = 0$$

$$\int_{-\infty}^{\infty} (a_2 + b_2 x + c_2 x^2) x e^{-\frac{x^2}{2A}} dx = 0 \quad (4')$$

$$\Phi_2(x) = c_2 (x^2 - A) e^{-\frac{x^2}{4A}}$$

Norm $\Phi_2(x)$

$$c_2^2 \int_{-\infty}^{\infty} (x^2 - A)^2 e^{-\frac{x^2}{2A}} dx \quad (5')$$

$$c_2 = \frac{1}{\sqrt{2\pi A}} \frac{1}{A} \frac{1}{\sqrt{2}}$$

$$c_2^2 = \frac{a_{ii}^2}{\sqrt{2\pi A}}$$

Φ_i Orthog, not normed

$$Q^{(0)}(\underline{\eta}) = \boxed{1}$$

$$Q_i^{(1)}(\underline{\eta}) = a_i \boxed{\eta_i} = a_i H_i^{(1)}(\eta_i)$$

$$Q_{ii}^{(2)}(\underline{\eta}) = a_{ii}[\eta_i^2 - A] = a_{ii} H_{ii}^{(2)}(\eta_i)$$

$$Q_{iii}^{(3)}(\underline{\eta}) = a_{iii}[\eta_i^3 - 3A\eta_i] = a_{iii} H_{iii}^{(3)}(\eta_i)$$

(3)

Now Hermite fns. $\rightarrow$ (orthonormal)

$$\phi_0(x) = \frac{1}{\sqrt[4]{2\pi A}}\left[1\right]e^{-\frac{x^2}{4A}}$$

$$\phi_1(x) = \frac{1}{\sqrt[4]{2\pi A}}\frac{1}{\sqrt{A}}\left[x\right]e^{-\frac{x^2}{4A}}$$

$$\phi_2(x) = \frac{1}{\sqrt[4]{2\pi A}}\frac{1}{A\sqrt{2}}\left[x^2-A\right]e^{-\frac{x^2}{4A}}$$

$$\phi_3(x) = \frac{1}{\sqrt[4]{2\pi A}}\frac{1}{A^{3/2}\sqrt{6}}\left[x^3-3Ax\right]e^{-\frac{x^2}{4A}}$$

Hermite polys $\rightarrow$ (orthonormal)

$$H_0(x) = [1]$$
$$H_1(x) = [x]$$
$$H_2(x) = [x^2-A]$$
$$H_3(x) = [x^3-3Ax]$$

$\{Q\}$ is what we want — want gen form.
Can get Q if all subscripts $=$; then Hermite poly.
Now req Q's to be symm. & each term must be symm in the subscripts $i, j, k, \ldots$; Also when $i=j=k=\ldots$ then must reduce to Hermite poly.

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(4)

$$H_n(x) = (-1)^n A^n e^{\frac{x^2}{2A}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2A}} \quad ; n = 0, 1, \ldots$$

$\frac{d^n}{dx^n} e^{-\frac{x^2}{2A}}$ is $\begin{cases} \text{even fn of } x \text{ if } n \text{ is even} \\ \text{odd} \quad \ldots \quad \ldots \quad \text{odd} \end{cases}$

$\frac{d^n}{dx^n} e^{-\frac{x^2}{2A}} = F(x) e^{-\frac{x^2}{2A}}$ ~~F~~ is $\begin{cases} \text{even if } n \text{ is even} \\ \text{odd} \ldots \ldots \text{odd} \end{cases}$

$$F(x) = F_e(x) + F_o(x)$$

$\frac{d^n}{dx^n} e^{-\frac{x^2}{2A}} = F_e(x) e^{-\frac{x^2}{2A}} + F_o(x) e^{-\frac{x^2}{2A}}$ even/odd if n even/odd

$$F_o(x) e^{-\frac{x^2}{2A}} = 0 \Rightarrow \cancel{F}$$

$\frac{d^n}{dx^n} e^{-\frac{x^2}{2A}} = F_e(x) e^{-\frac{x^2}{2A}}$, n even.

$$H_n(x) = (-1)^n A^n e^{\frac{x^2}{2A}} F_e(x) e^{-\frac{x^2}{2A}}$$

$$= (-1) A^n F_e(x) \quad , \quad n = 0, 2, 4, \ldots$$

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Gen expression for Q-polys

$$Q^{(2)}_{ij}(\underline{\eta}) = a_{ij}(\eta_i \eta_j - A\delta_{ij})$$

Recall got this from $\overline{\eta_i \eta_j} = A\delta_{ij}$

so can equiv write $Q^{(2)}_{ij}(\underline{\eta}) = a_{ij}(\eta_i\eta_j - \overline{\eta_i \eta_j})$

$$Q^{(3)}_{ijk}(\underline{\eta}) = a_{ijk}\left[\eta_i\eta_j\eta_k - A(\eta_i\delta_{jk} + \eta_j\delta_{ik} + \eta_k\delta_{ij})\right]$$

get 2nd term by symm const of terms $\eta_i\overline{\eta_j\eta_k} = \eta_i A\delta_{jk}$

so get $A(\eta_i\delta_{jk} + \eta_j\delta_{ki} + \eta_k\delta_{ij})$ as 2nd term.

these must reduce to $H_2(x)$ & $H_3(x)$ for $i=j=k$.

and we have an exp for $H_n(x)$.

$$Q^{(4)}_{ijkl}(\underline{\eta}) = a_{ijkl}\left[\eta_i\eta_j\eta_k\eta_l - A(\eta_i\eta_j\delta_{kl} + \eta_j\eta_k\delta_{il} + \dots) + A^2(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\right]$$

$\eta_i\eta_j\overline{\eta_k\eta_l} = \eta_i\eta_j A\delta_{kl}$

etc. (6 ways) (symm)

$\overline{\eta_i\eta_j}\,\overline{\eta_k\eta_l} = A^2\delta_{ij}\delta_{kl}$

etc. (3 ways) to form avg of pairs

must reduce to

$H_4(x) = x^4 - 6Ax^2 + 3A^2$

$$Q^{(4)}_{iiii}(\underline{\eta}) = a_{iiii}\left[\eta_i^4 - 6A\eta_i^2 + 3A^2\right]$$

Props of Q fns:

$$\overline{Q^{(m)}_{ijk\dots}(\underline{\eta})\, Q^{(n)}_{ijk\dots}(\underline{\eta})} = 0, \quad n \neq m$$

$\left\{ \begin{array}{l} i_1, j_1, k_1, \dots \text{ can be same} \\ \text{or diff or } i_2, j_2, k_2, \dots \end{array} \right.$

$Q^{(n)}_{ijk\dots}(\underline{\eta})$ is orthog all consts $\Rightarrow$ $\overline{Q^{(n)}_{ijk\dots}(\underline{\eta})} = 0$, $\underline{n \geq 1}$

(2)

product of fns of same degree

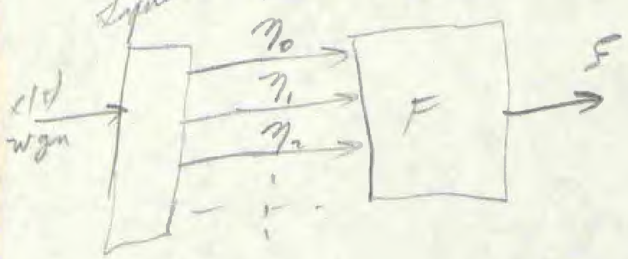
$$\overline{Q_i^{(1)}(\underline{\eta})\, Q_j^{(1)}(\underline{\eta})} = a_i a_j \overline{\eta_i \eta_j} = a_i a_j \delta_{ij} = 0, \quad i \neq j$$

$$\overline{Q_{ij}^{(2)}(\underline{\eta})\, Q_{kl}^{(2)}(\underline{\eta})} = \overline{Q_{ij}^{(2)}(\underline{\eta})}\; \overline{Q_{kl}^{(2)}(\underline{\eta})} = 0 \quad \{ i,j \neq k,l \text{ unordered}$$

$$\overline{Q_{ijk\ldots}^{(n)}(\underline{\eta})\, Q_{abc}^{(m)}(\underline{\eta})} = 0 \quad \text{if no } i,j,k = \text{one of } a,b,c \ldots$$

i.e., no η_i appears in both Q fns.

Synonyms: How try to find consts a_i, a_{ij}, . . . for given syst !



η_0, η_1, η_2, . . .

$\eta_0\eta_0$, $\eta_0\eta_1$, $\eta_1\eta_1$, $\eta_0\eta_2$, . . .

$\eta_0\eta_0\eta_0$, . . .

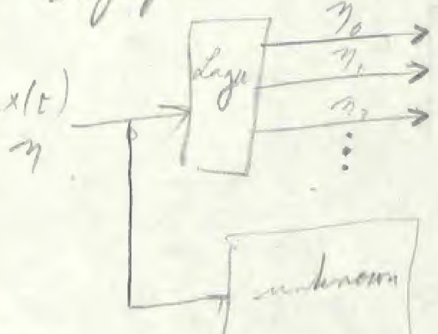
all have been orthog by Q fns.

$$\xi = \sum_0^\infty {}_i\, Q_i^{(1)}(\underline{\eta}) + \sum_0^\infty {}_i \sum_0^\infty {}_j\, Q_{ij}^{(2)}(\underline{\eta}) + \ldots$$

$$= F_1(\underline{\eta}) + F_2(\underline{\eta}) + \ldots$$

$$= \sum_1^\infty {}_n\, F_n(\underline{\eta})$$

in this model, non lin syst can vary only in coeffs of amplifiers, etc in n.l. box.



$$F_1(\underline{\eta}) = \sum_0^\infty {}_i\, Q_i^{(1)}(\underline{\eta}) = \sum_0^\infty {}_i\, a_i \eta_i = \sum_0^\infty {}_i\, a_i H_i^{(1)}(\underline{\eta})$$

$H_i^{(1)}(\underline{\eta})$ is m.d. Hermite poly.

(3)

Want to find just $F_1(\underline{\eta})$ for the unknown syst:

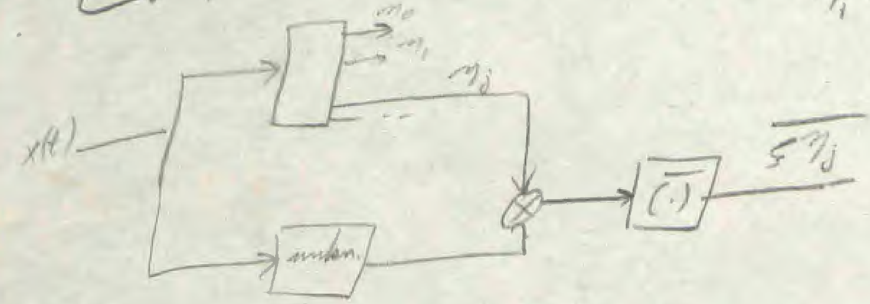
pick j = output η_j

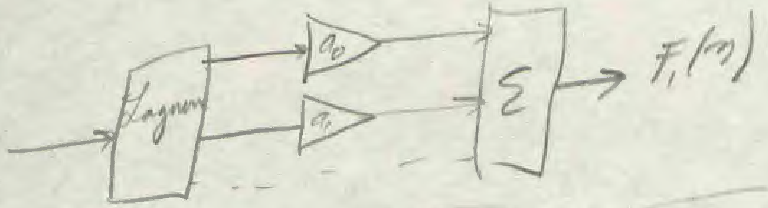
find $\overline{\xi \eta_j} = \sum_{n=1}^{\infty} \overline{F_n(\underline{\eta})\eta_j} = \overline{F_1(\underline{\eta})\eta_j} = \sum_{i=0}^{\infty} a_i \overline{\eta_i \eta_j} = a_j A$

η_j orthog all F_n, $n>1$

so $\overline{\xi \eta_j} = a_j A$

or $a_j = \frac{1}{A}\overline{\xi \eta_j}$





2^{nd} term, what is $F_2(\underline{\eta})$?:

$F_2(\underline{\eta}) = \sum_i \sum_j a_{ij}(\eta_i\eta_j - A\,\delta_{ij})$

Suppose for illust $i = 0, 1, 2$; $j = 0, 1, 2$.

$$\cancel{F(\eta)} \quad \begin{bmatrix} \eta_0\eta_0 - A & \eta_0\eta_1 & \eta_0\eta_2 \\ \eta_1\eta_0 & \eta_1\eta_1 - A & \eta_1\eta_2 \\ \eta_2\eta_0 & \eta_2\eta_1 & \eta_2\eta_2 - A \end{bmatrix}$$

To get a_{00},

find $\overline{\xi(\eta_0\eta_0 - A)} = \sum_{n=1}^{\infty} \overline{F_n(\underline{\eta})[\eta_0\eta_0 - A]} = \overline{\sum_i\sum_j a_{ij}^{(2)}(\underline{\eta})[\eta_0^2 - A]}$ — see over page 80

$= a_{00}\overline{(\eta_0^2 - A)^2} = a_{00}[\overline{\eta_0^4} - 2A\overline{\eta_0^2} + A^2]$

$= a_{00}[3A^2 - 2A^2 + A^2] = 2A^2 a_{00}$

so $a_{00} = \frac{1}{2A^2}\overline{\xi(\eta_0^2 - A)}$.

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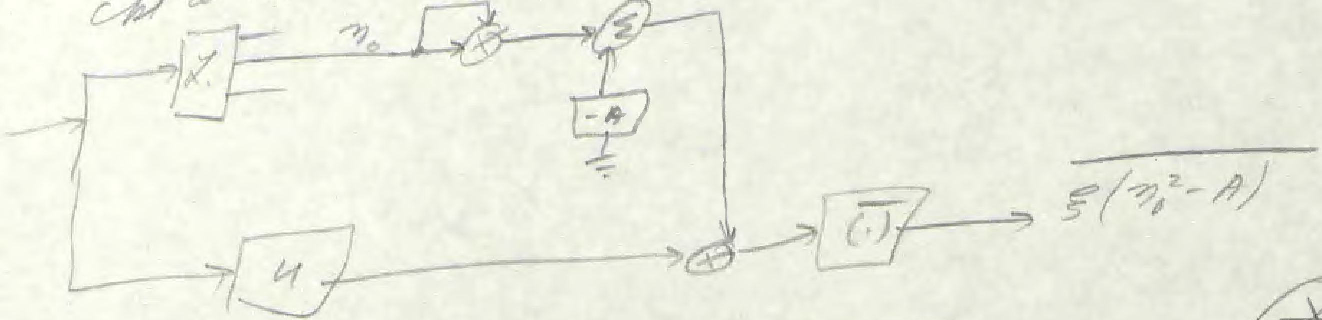
(4) ckt to do this:



Go back to $\otimes$ $\overline{(\eta_i \eta_j - A\delta_{ij})(\eta_0^2 - A)}$

$= \overline{\eta_i \eta_j \eta_0^2} - A\,\overline{\eta_i \eta_j} = 0$ if $i \neq j$, $i = 0$ or not.

$\neq 0$ value only when $i = 0 = j$

sim $a_{ii} = \frac{1}{2A^2}\overline{\xi(\eta_i^2 - A)}$

go on next time.

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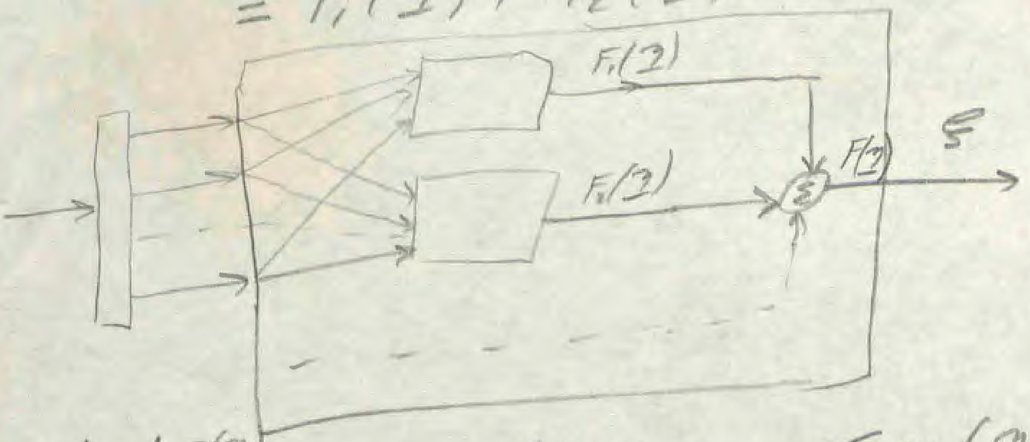
$$\xi = \sum_i Q_i^{(1)}(\eta) + \sum_i \sum_j Q_{ij}^{(2)}(\eta) + \dots$$

$$= F_1(\eta) + F_2(\eta) + \dots$$



Prev found $F_1(\eta)$

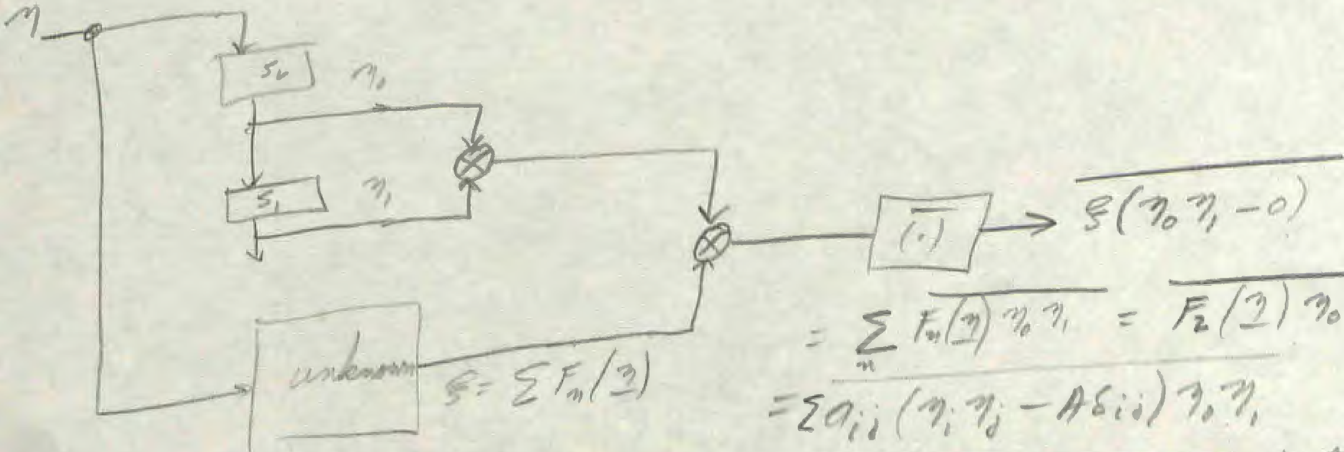
Now $F_2(\eta) = \sum_i \sum_j Q_{ij}^{(2)}(\eta) = \sum_i \sum_j a_{ij}(\eta_i \eta_j - A\delta_{ij})$

$i, j = 0, 1, 2$

Can write $(a_{ij}) = \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ - & - & - \\ - & - & a_{22} \end{bmatrix} \phi(\eta_{ij}) \phi (\delta_{ij})$

Found a_{00} prev

Find a_{01}:



$$= \sum_n \overline{F_n(\eta)\eta_0\eta_1} = \overline{F_2(\eta)\eta_0\eta_1}$$

$$= \sum a_{ij} \overline{(\eta_i \eta_j - A\delta_{ij})\eta_0\eta_1}$$

$$= (a_{10} + a_{01}) \overline{(\eta_0 \eta_1 - 0)\eta_0\eta_1}$$

by prev proof or :: proof to be given when look at 3rd order SSE.

$$= 2a_{01} \overline{\eta_0^2 \eta_1^2} = a_{01} A^2$$

$$\boxed{a_{01} = \frac{\overline{\xi \eta_0 \eta_1}}{2A^2}} \Longleftarrow$$

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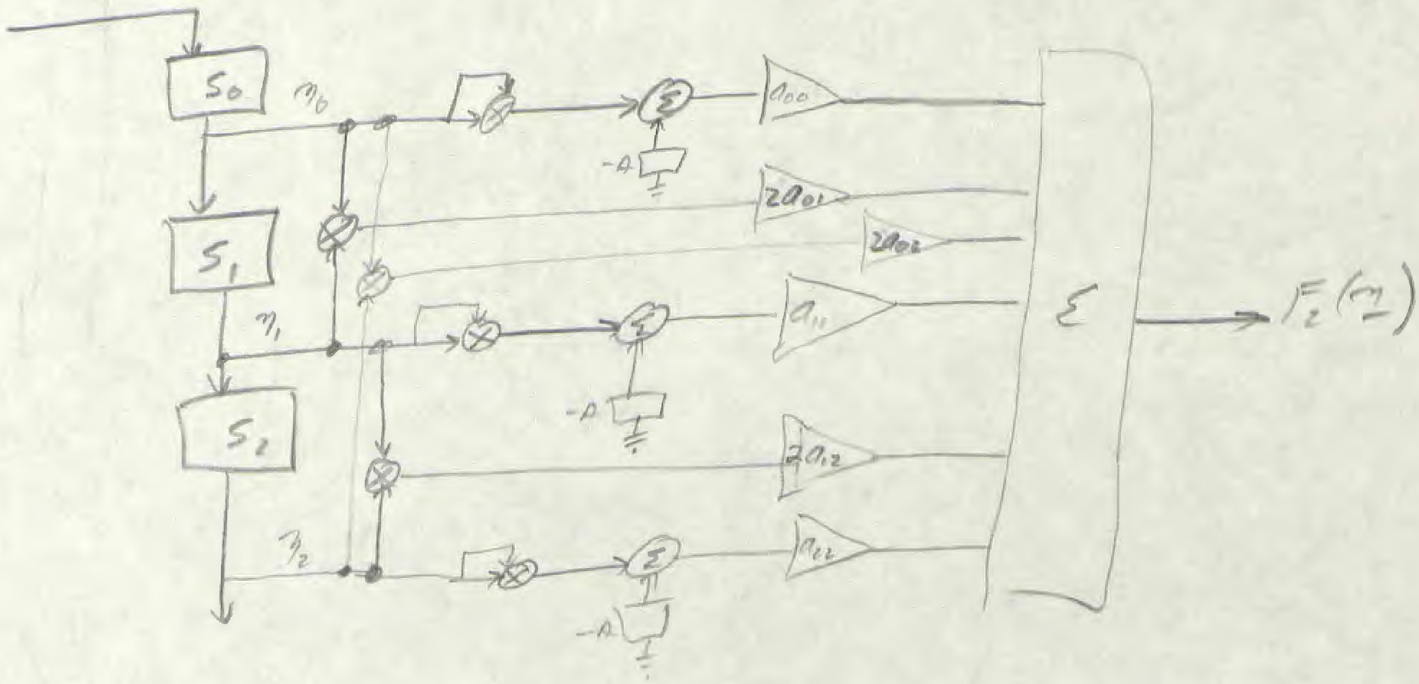
(2)

Note $a_{10} = a_{01}$ since ord of mult unimport.

& then see $a_{ij} = a_{ji}$

Now we have found a_{ij}, build syst model;



Now look at 3rd order term: $F_3(\underline{\eta}) = \sum_i \sum_j \sum_k Q_{ijk}^{(3)}(\underline{\eta})$

$$\overline{Q_{ijk}^{(3)}(\underline{\eta})\, Q_{lmn}^{(3)}(\underline{\eta})} = \overline{Q_{ijk}^{(3)}(\underline{\eta})\, \eta_l \eta_m \eta_n}$$

since $Q_{ijk}^{(3)}(\underline{\eta})$ is orthog all lower deg ~~pad~~ polys in the η's.

Consid $i \neq l$, $j = m$, $k = n$.

$$\overline{Q_{ijk}^{(3)}(\underline{\eta})\, \eta_l\, \eta_j\, \eta_k} = \overline{\eta_l}\; \overline{Q_{ijk}^{(3)}(\underline{\eta})\, \eta_j\, \eta_j} = 0$$

Consid $i \neq l$, $j \neq m$, $k = n$

$$\overline{Q_{ijk}^{(3)}(\underline{\eta})\, \eta_l\, \eta_m\, \eta_k} = \overline{\eta_l \eta_m}\; \overline{Q_{ijk}^{(3)}(\underline{\eta})\, \eta_k} = 0 \text{ since } Q \text{ orthog } \eta_k$$

~~$i \neq l$, $j = m$~~ When no $i, j, k = l, m, n$ get zero [proved prev.]

(3)

Thae only term left is ~~when all~~ ~~...~~
when i, j, k & l m n are only 3 distinct ...
This can be extended to n^{th} order
& reverted to 2^{nd} order where we referred to
the proof.

$$a_{ijk}^{(3)}(\eta) = a_{ijk}\, H_{ijk}^{(3)}(\eta)$$

$$\overline{\left[H_{ijk}^{(3)}(\eta)\right]^2} = \overline{\left[\eta_i \eta_j \eta_k - A(\eta_i \delta_{jk} + \eta_j \delta_{ik} + \eta_k \delta_{ji}\right] \eta_i \eta_j \eta_k}$$

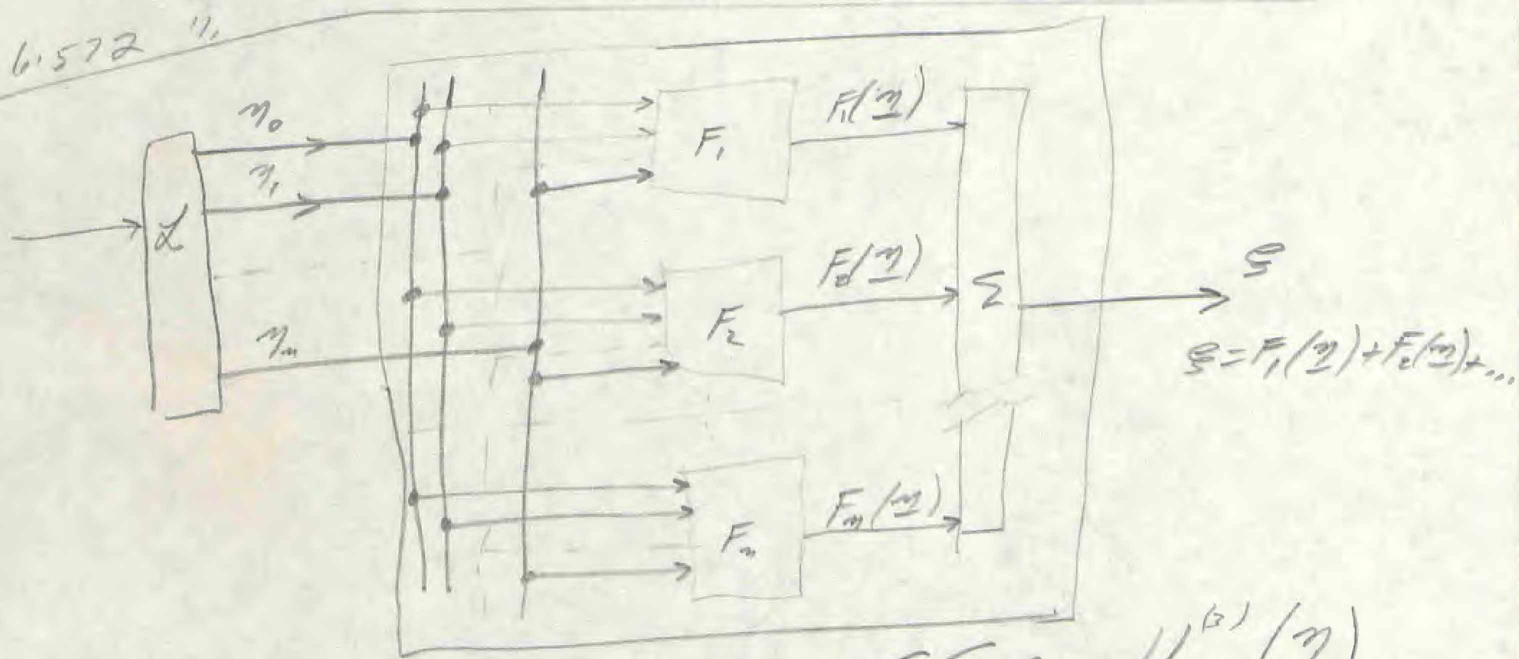
$$= \overline{\eta_i^2 \eta_j^2 \eta_k^2} - A\,\overline{\eta_i^2 \eta_j \eta_k \delta_{jk}} - A\,\overline{\eta_i \eta_j^2 \eta_k \delta_{ik}} + A\,\overline{\eta_i \eta_j \eta_k^2 \delta_{ji}}$$

if $i \neq j = k$; or $i = j \neq k$, or $i = k \neq j$

$$\overline{[\ \ ]^2} = 2A^3 \quad ; \quad \cancel{\text{if } i = j \neq k,\ \overline{[\ \ ]^2} = 2A}$$

if $i = j = k$; $\overline{[\ \ ]^2} = 3!\,A^3$

6.572 η_i



Consid $F_3(\underline{\eta}) = \sum_i \sum_j \sum_k Q^{(3)}_{ijk}(\underline{\eta}) = \sum\sum\sum a_{ijk} H^{(3)}_{ijk}(\underline{\eta})$

$= \sum\sum\sum a_{ijk}\left[\eta_i \eta_j \eta_k - A(\eta_i \delta_{jk} + \eta_j \delta_{ki} + \eta_k \delta_{ij})\right]$

Consid $i, j, k = 0, 1$

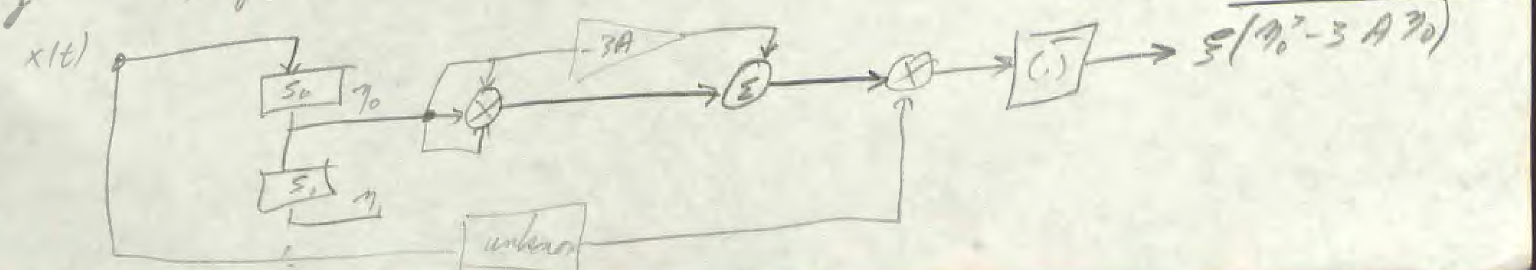
$i, j, k = 0,0,0 ;\ 0,0,1 ;\ 0,1,0 ;\ 0,1,1 ;\ \ldots\ ;\ 1,1,1$ } 16 ways

~~Consid~~ 0,0,0 ; $a_{000}[\eta_0^3 - 3A\eta_0]$

0,0,1 ; $a_{001}[\eta_0^2 \eta_1 - A\eta_1]$

0,1,0 ; etc.

To get a_{000}, form $[\eta_0^3 - 3A\eta_0]$



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(2)

$$\overline{\xi(\eta_0^3-3A\eta_0)} = \overline{\sum F_n(\underline{\eta})(\eta_0^3-3A\eta_0)}$$

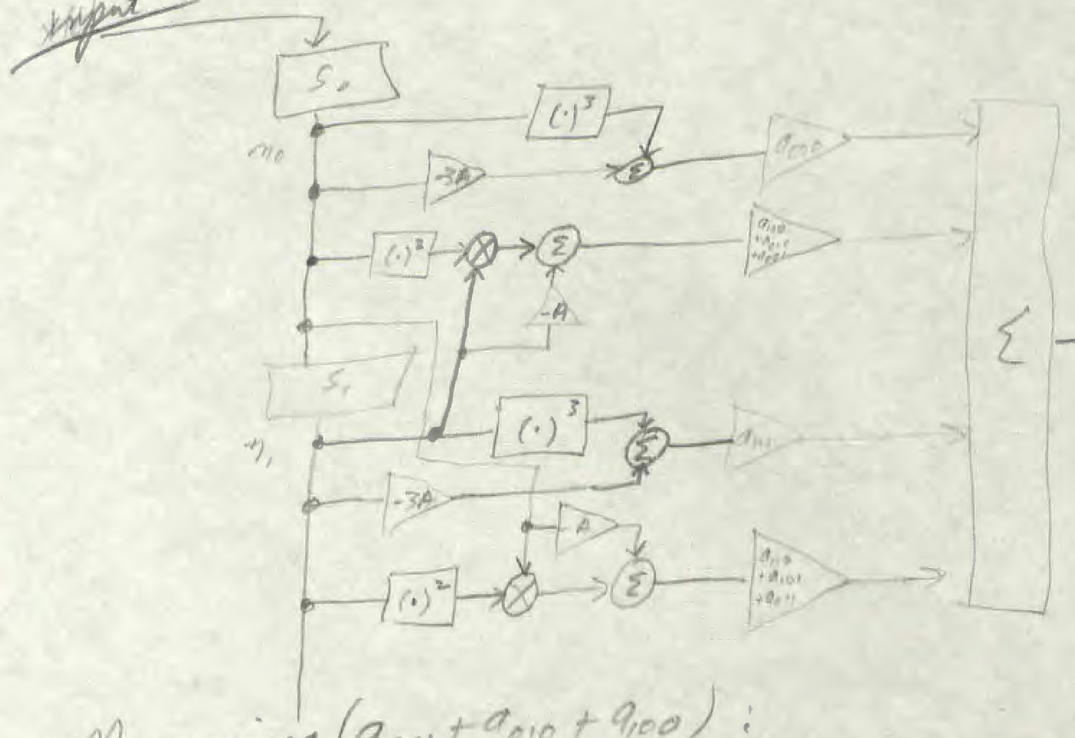
$$= \overline{F_3(\underline{\eta})(\eta_0^3-3A\eta_0)} =$$

$$= a_{000}\,\overline{(\eta_0^3-3A\eta_0)^2}$$

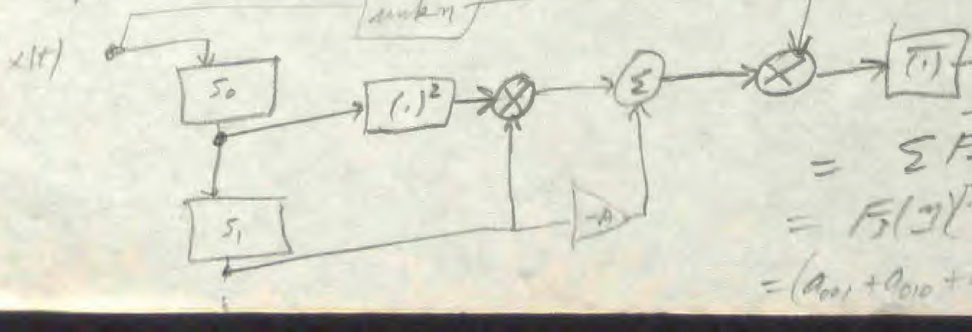
This is just the mean sq of one of the H's;

$$= a_{000}\,3!\,A^3 \quad \text{or} \quad a_{000} = \frac{1}{3!\,A^3}\,\overline{\xi(\eta_0^3-3A\eta_0)}$$

Synthesis of $F_3(\underline{\eta})$:



Measuring $(a_{001}+a_{010}+a_{100})$:



$$\rightarrow \overline{\xi(\eta_0^2\eta_1-A\eta_1)}$$

$$= \overline{\sum F_n(\underline{\eta})(\eta_0^2\eta_1-A\eta_1)}$$

$$= \overline{F_3(\underline{\eta})(\eta_0^2\eta_1-A\eta_1)}$$

$$= (a_{001}+a_{010}+a_{100})\,\overline{(\eta_0^2\eta_1-A\eta_1)^2} =$$

$$= (a_{001}+\cdots)\,2A^3$$

$$\text{or } (a_{001}+\cdots) = \frac{1}{2A^3}\,\overline{\xi(\eta_0^2\eta_1-A\eta_1)}$$

$$= 3a_{001}$$

$$\text{; Sim } (a_{110}+a_{011}+a_{101}) = \frac{1}{2A^3}\,\overline{\xi(\eta_0\eta_1^2-A\eta_0)}$$

(3)

Compar of G fn w/ ~~present~~ approach [F fn]:

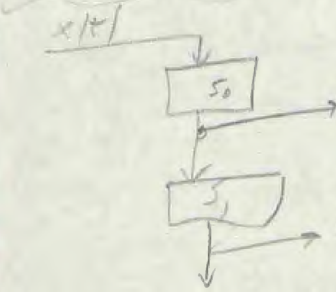
Compar of G fn & Hermite polys:

$$y(t) = \sum_{n=1}^{\infty} G_n[h_n, x(t)]$$

$$G_1[h_1, x(t)] = \int h_1(\tau)\, x(t-\tau_1)\, d\tau_1$$

$$G_2[h_2, x(t)] = \iint h_2(\tau_1,\tau_2)\, x(t-\tau_1)\, x(t-\tau_2)\, d\tau_1 d\tau_2 - A\int h_2(\tau_1,\tau_1)\, d\tau_1$$

$$h_1(\tau_1) = \sum_{m=0}^{\infty} c_m\, l_m(\tau_1)$$



$$v_0(t) = \int l_0(\tau)\, x(t-\tau)\, d\tau$$

$$v_1(t) = \int l_1(\tau)\, x(t-\tau)\, d\tau$$

$$G_1[h_1, x(t)] = \int h_1(\tau_1)\, x(t-\tau_1)\, d\tau_1 = \int \sum_{m=0}^{\infty} c_m\, l_m(\tau_1)\, x(t-\tau_1)\, d\tau_1 = \sum_{m=0}^{\infty} c_m\, v_m(t)$$

$$\underline{v}(t) = v_0(t), v_1(t), \ldots$$

$$G_1(\underline{v}(t)) = \sum_{m=0}^{\infty} c_m\, v_m(t)$$

$$\xi = \sum_{n=1}^{\infty} F_n(\underline{v})$$

$$F_1(\underline{v}) = \sum_{i=0}^{\infty} a_i\, v_i$$

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(4)

2^{nd} order $h_2(\tau_1, \tau_2) = \sum_{m_1} \sum_{m_2} C_{m_1 m_2} l_{m_1}(\tau_1) l_{m_2}(\tau_2)$

$\iint h_2(\tau_1,\tau_2)\, x(t-\tau_1)\, x(t-\tau_2)\, d\tau_1\, d\tau_2$

$= \iint \left[\sum\sum - - -\right] x(t-\tau_1)\, x(t-\tau_2)\, d\tau_1\, d\tau_2$

$= \sum_{m_1} \sum_{m_2} C_{m_1 m_2}\, v_{m_1}(t)\, v_{m_2}(t)$

Now compute ~~1st~~ const term:

$\int h_2(\tau_1,\tau_1)\, d\tau_1 = \int \sum_{m_1} \sum_{m_2} C_{m_1 m_2}\, l_{m_1}(\tau_1)\, l_{m_2}(\tau_1)\, d\tau_1$

$= \sum_{m_1} \sum_{m_2} C_{m_1 m_2}\, \delta_{m_1 m_2}$

So $G_2\left[\underline{v(t)}\right] = \sum_{m_1} \sum_{m_2} C_{m_1 m_2} \left[v_{m_1}(t)\, v_{m_2}(t) \right] - A\, \delta_{m_1 m_2} \Big)$

$\updownarrow$

$F_2(\eta) = \sum_i \sum_j a_{ij} \left[\eta_i \eta_j - A\, \delta_{ij} \right]$

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Continue comparison of G's + F's

3rd order: $G_3[h_3, x(t)] = \iiint h_3(\tau_1,\tau_2,\tau_3)\,x(t-\tau_1)\,x(t-\tau_2)\,x(t-\tau_3)\,d\tau_1\,d\tau_2\,d\tau_3$
$\qquad - 3A \iint h_3(\tau_1,\tau_2,\tau_2)\,x(t-\tau_1)\,d\tau_1\,d\tau_2$

$$h_3(\tau_1,\tau_2,\tau_3) = \sum_{m_1}\sum_{m_2}\sum_{m_3} c_{m_1 m_2 m_3}\, l_{m_1}(\tau_1)\, l_{m_2}(\tau_2)\, l_{m_3}(\tau_3)$$

$$\iiint h_3(\ )\,x(\ )\,x(\ )\,x(\ )\,d\tau_{123} = \iiint \sum\sum\sum c_{m_1 m_2 m_3}\, l_{m_1}(\ )\, l_{m_2}(\ )\, l_{m_3}(\ )\, x(\ )\, x(\ )\, x(\ )\, d\tau_{123}$$

$$= \sum\sum\sum c_{m_1 m_2 m_3}\, v_{m_1}(t)\, v_{m_2}(t)\, v_{m_3}(t)$$

(where $v_m(t) = \int l_m(\tau)\,x(t-\tau)\,d\tau$)

$$\iint h_3(\tau_1,\tau_2,\tau_2)\,x(t-\tau_1)\,d\tau_1\,d\tau_2 = \iint \sum\sum\sum c_{m_1 m_2 m_3}\, l_{m_1}(\tau_1)\, l_{m_2}(\tau_2)\, l_{m_3}(\tau_2)\, x(t-\tau_1)\, d\tau_1\, d\tau_2$$

$$= \sum_{m_1}\sum_{m_2}\sum_{m_3} c_{m_1 m_2 m_3}\, v_{m_1}(t)\,\delta_{m_2 m_3} = \sum\sum\sum c_{m_1 m_2 m_3}\, v_{m_3}(t)\,\delta_{m_1 m_2} = \sum\sum\sum c_{m_1 m_2 m_3}\, v_{m_2}(t)\,\delta_{m_1 m_3}$$

by symm of kernel

$$= \frac{1}{3}\left[\ (1) + (2) + (3)\ \right]$$

so $$G_3[v(t)] = \sum\sum\sum c_{m_1 m_2 m_3}\left[v_{m_1}(t)\,v_{m_2}(t)\,v_{m_3}(t) - A\left(v_{m_1}(t)\,\delta_{m_2 m_3} + v_{m_2}(t)\,\delta_{m_1 m_3} + v_{m_3}(t)\,\delta_{m_1 m_2}\right)\right]$$

c.f.
$$F_3(\eta) = \sum_i\sum_j\sum_k a_{ijk}\left[\eta_i\,\eta_j\,\eta_k - A\left(\eta_i\,\delta_{jk} + \eta_j\,\delta_{ki} + \eta_k\,\delta_{ij}\right)\right]$$

2nd term of F_3 found by avg $\eta_i\,\eta_j\,\eta_k$ pair at a time + mkg symm

" " G_3 " " $\iiint h_3(\ )\,x(t-\tau_1)\,x(t-\tau_2)\,x(t-\tau_3)\,d\tau_1\,d\tau_2\,d\tau_3$

must mult by 3 to acct for fact that this is already symm. (?)

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2

Now look at $G_4[v(t)]$

$$= \sum\sum\sum\sum c_{m_1 m_2 m_3 m_4} \Big\{ v_{m_1}(t) v_{m_2}(t) v_{m_3}(t) v_{m_4}(t)$$

avg 1 pair at a time $\rightarrow$ $$- A\Big[v_{m_1}(t) v_{m_2}(t) \delta_{m_3 m_4} + \underbrace{v_{m_1}(t) v_{m_3}(t) \delta_{m_2 m_4} + \cdots}_{6 \text{ terms}} \Big]$$

avg 2 pairs at a time $\rightarrow$ $$+ A^2 \Big[\delta_{m_1 m_2} \delta_{m_3 m_4} + \delta_{m_1 m_3} \delta_{m_2 m_4} + \delta_{m_1 m_4} \delta_{m_2 m_3} \Big] \Big\}$$

Sim $F_4(\underline{T})$

$$G_4[h_4, x(t)] = \iiiint h_4(\tau_1, \tau_2, \tau_3, \tau_4)\, x(\,)x(\,)x(\,)x(\,)\, d\underline{\tau}$$

$$- 6 \iiiint h_4(\underline{\tau})\, \underline{x(t-\tau_1) x(t-\tau_2)}\, \overline{x(t-\tau_3) x(t-\tau_4)}\, \underline{d\tau}$$

$$+ 3 \iiiint h_4(\underline{\tau})\, \overline{x(t-\tau_1) x(t-\tau_2)}\, \overline{x(t-\tau_3) x(t-\tau_4)}\, d\underline{\tau}$$

$$A \iiint h_4(\tau_1, \tau_2, \tau_3, \tau_3)\, x(t-\tau_1) x(t-\tau_2)\, d\tau_1\, d\tau_2\, d\tau_3$$

$$A \iiint h_4(\tau_1, \tau_2, \tau_3, \tau_4)\, \overline{x(t-\tau_1) x(t-\tau_2)}\, d\tau_1\, d\tau_2\, d\tau_3$$

$$= A^2 \iint h_4(\tau_1, \tau_1, \tau_3, \tau_3)\, d\tau_1\, d\tau_3$$

(3) Now look at 5th ord

$$F_5(\eta) = \sum\sum\sum\sum\sum a_{ijklm}\Big[\eta_i\eta_j\eta_k\eta_l\eta_m$$

$$-A\Big(\underbrace{\eta_i\eta_j\eta_k\delta_{lm} + \eta_i\eta_j\eta_l\delta_{km} + }_{10\ \text{terms}}$$

$$+A^2\Big(\underbrace{\eta_i\delta_{jk}\delta_{lm} + \eta_i\delta_{jl}\delta_{km} + \cdots}_{15\ \text{terms}}$$

~~so~~ $\cancel{F_5(\eta) = \sum\sum\sum\sum\sum a_{ijklm}\Big[\eta_i\eta_j\eta_k\eta_l\eta_m -}$

So

$$G_5[h_5, x(t)] = \int\int\int\int\int h_5(\tau)\, x(\,)x(\,) \ldots d\tau$$

$$\int\int\int\int\int h_5(\tau_1,\tau_2,\tau_3,\tau_4,\tau_5)\, x(t-\tau_1)x(t-\tau_2)x(t-\tau_3)\,d\tau_1 d\tau_2 d\tau_3 d\tau_4$$

$$-10A\int\int\int\int h_5(\tau_1,\tau_2,\tau_3,\tau_4,\tau_4)\, x(t-\tau_1)x(t-\tau_2)x(t-\tau_3)\,d\tau_1 d\tau_2 d\tau_3 d\tau_4$$

$$+15A^2\int\int\int h_5(\tau_1,\tau_2,\tau_2,\tau_4,\tau_4)\, x(t-\tau_1)\,d\tau_1 d\tau_2 d\tau_4$$

$\cancel{10 = \frac{5!}{3!\,2!} = \frac{5 \cdot 4}{2}}$

$\cancel{10\ \frac{3!}{2!} = 30}$

from table for $H_5(x) = x^5 - 10Ax^3 + 15A^2x$

April 28

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Cov G+F cont:

$$H_n(x) = x^n - 1C_2^n A x^{n-2} + 1\cdot 3 C_4^n A^2 x^{n-4} - 1\cdot 3\cdot 5\, C_6^n A^3 x^{n-6} + \cdots$$

C_m^n = # comb of n things taken m at a time.

$$= \frac{n!}{m!(n-m)!}$$

Recall in avging Gauss r.v.'s, had N = # of distinct ways of partitioning $2m$ things into pairs

$$N = \frac{(2m)!}{m!\,2^m}$$

$1\cdot 3$ = # of distinct ways of partitioning 4 things into pairs

$1\cdot 3\cdot 5$ = 6

Consid $H_4(x) = x^4 - C_2^4 A x^2 + 1\cdot 3\ C_4^4 A^2$

$F_4(\underline{\eta})$: 1st term: $\eta_i \eta_j \eta_k \eta_l$

2nd term: $\overline{\eta_i \eta_j}\, \eta_k \eta_l$, $\overline{\eta_i \eta_k}\, \eta_j \eta_l$, 6 terms

$$C_2^4 = \frac{4!}{2!\,2!} = 6$$

3rd term: $\overline{\eta_i \eta_j}\, \overline{\eta_k \eta_l}$, $\overline{\eta_i \eta_k}\, \overline{\eta_j \eta_l}$, ... 3 terms

$$\left.\begin{array}{l} C_4^4 = 1 \\ 1\cdot 3 = 3 \end{array}\right\} C_4^4 (1\cdot 3) = 3$$

$H_5(x) = x^5 - 1C_2^5 A x^3 + 1\cdot 3\, C_4^5 x$

$F_5(\underline{\eta})$; 2nd term $\overline{\eta_i \eta_j}\, \eta_k \eta_l \eta_m$, $\overline{\eta_i \eta_k}\, \eta_j \eta_l \eta_m$, ... 10 terms

$C_2^5 = \frac{5!}{2!\,3!} = 10$ terms ✓

3rd term $\overline{\eta_i \eta_j}\, \overline{\eta_k \eta_l}\, \eta_m$, $\overline{\eta_i \eta_k}\, \overline{\eta_j \eta_l}\, \eta_m$, ...

$(1\cdot 3)\, C_4^5 = (1\cdot 3)\frac{5!}{4!} = 15$ terms ✓

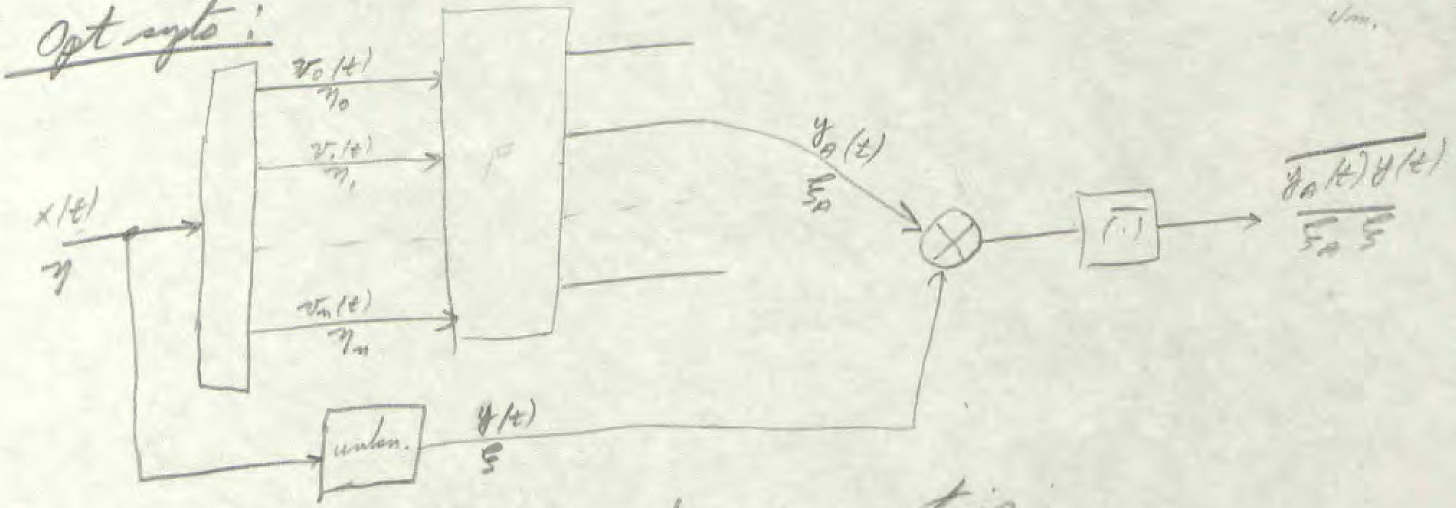
15 terms $\Big\}$ $\eta_i \eta_j \eta_k \eta_l \eta_m$, ... 5 ways each one has 3 ways of pairs: $\overline{\eta_i \eta_j}\,\overline{\eta_k \eta_l}\, \eta_m + \overline{\eta_i \eta_k}\,\overline{\eta_j \eta_l}\, \eta_m$ + ...

88

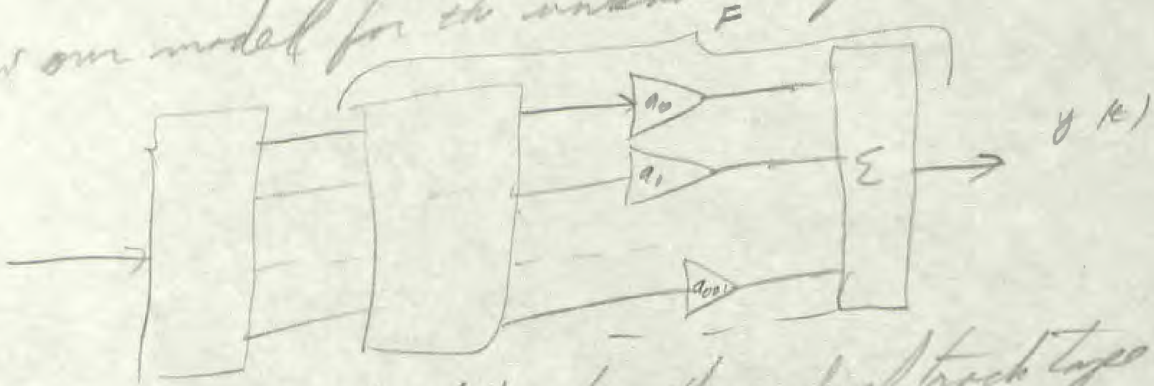
2

Mar 16 No. 13

Opt systs:



Now our model for the unknown syst is:



Unk syst need not be phys. thus, dual track tape rec of inp & outp is suff.

Let ϕ_m be $\ni$ $\int_{-\infty}^{\infty} \phi_m(x)\phi_n(x)\,dx = \delta_{mn}$

$$f(x) = \sum_1^{\infty} a_n \phi_n(x)$$

$$a_m = \int_{-\infty}^{\infty} f(x)\phi_m(x)\,dx$$

Integral sq error is $\int_{-\infty}^{\infty}\left[f(x) - \sum_1^{\infty} a_n \phi_n(x)\right]^2 dx$

$$= \int_{-\infty}^{\infty} f^2(x)\,dx - 2\sum_{n=1}^{\infty} a_n \int_{-\infty}^{\infty} f(x)\phi_n(x)\,dx + \int \sum_n a_n \phi_n(x) \sum_m a_m \phi_m(x)\,dx$$

$$= \int_{-\infty}^{\infty} f^2(x)\,dx + \sum\left[a_n - \int_{-\infty}^{\infty} f(x)\phi_n(x)\,dx\right]^2 - \sum\left[\int_{-\infty}^{\infty} f(x)\phi_n(x)\,dx\right]^2$$

(3)

Want min integral of error

a_n appears only in central term & this term always ≥ 0

to min, set $= 0 \Rightarrow a_n = \int_{-\infty}^{\infty} f(x)\, \phi_n(x)\, dx$

~~This is how we find the coeff~~

Multi-dim case:

$\Rightarrow$ normalized H.

$$\xi = \sum_n \sum_{\alpha(n)} a^{(n)}_{\alpha(n)} H^{(n)*}_{\alpha(n)}(\underline{\eta})$$

$\alpha(1) = i$

$\alpha(2) = i, j$

$\vdots$

$$\overline{H^{(n)*}_{\alpha(n)}(\underline{\eta}) H^{(m)*}_{\alpha(m)}(\underline{\eta})} = \delta_{mn}$$

$$\overline{\xi H^{(m)*}_{\alpha(m)}(\underline{\eta})} = \sum_n \sum_{\alpha(n)} a^{(n)}_{\alpha(n)} \overline{H^{(n)*}_{\alpha(n)}(\underline{\eta}) H^{(m)*}_{\alpha(m)}(\underline{\eta})}$$

$$= a^{(m)}_{\alpha(m)}$$

Error: $$\overline{\mathcal{E}^2} = \overline{\left[\xi_d - \sum_n \sum_{\alpha(n)} a^{(n)}_{\alpha(n)} H^{(n)*}_{\alpha(n)}(\underline{\eta})\right]^2}$$

$$= \overline{\xi_d^2} + \sum_n \sum_{\alpha(n)} \left[a^{(n)}_{\alpha(n)} - \overline{\xi_d H^{(n)*}_{\alpha(n)}(\underline{\eta})}\right]^2 - \sum_n \sum_{\alpha(n)} \left[\overline{\xi_d H^{(n)*}_{\alpha(n)}(\underline{\eta})}\right]^2$$

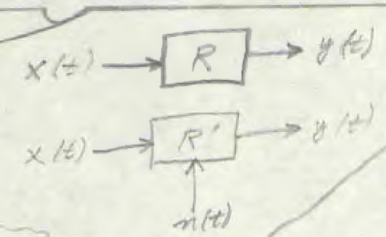
min. $\overline{\mathcal{E}^2}$ when $a^{(n)}_{\alpha(n)} = \overline{\xi_d H^{(n)*}_{\alpha(n)}(\underline{\eta})}$

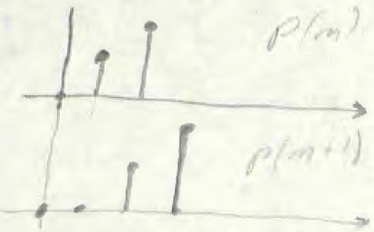
$\overline{\mathcal{E}^2_{min}} =$ [illegible]

Exactly what we did to determine our coefficients; so we minimized mse when we formed our model.

3 May 61

6.5 72





$\frac{\rho_{11}-\gamma}{\delta} \to \mu''$

Nonlinear no memory systs:

Hermite fns: $\phi_n(x) = \frac{1}{\sqrt[4]{2\pi A}\,\sqrt{n!}\,A^{n/2}} H_n(x)\, e^{-\frac{x^2}{4A}}$, $n = 0, 1, \ldots$

Hermite poly

$$\int_{-\infty}^{\infty} \phi_n(x)\,\phi_m(x)\,dx = \delta_{mn}$$

$$f(x) = \sum_{n=0}^{\infty} b_n \phi_n(x) \quad ; \quad b_n = \int_{-\infty}^{\infty} f(x)\,\phi_n(x)\,dx$$

$$\frac{1}{\sqrt{2\pi A}} \int_{-\infty}^{\infty} H_m(x) H_n(x)\, e^{-\frac{x^2}{2A}}\,dx = A^n n!\, \delta_{mn}$$

$$f(x) = \sum_{0}^{\infty} a_m H_m(x)$$

$$\frac{1}{\sqrt{2\pi A}} \int_{-\infty}^{\infty} f(x) H_m(x)\, e^{-\frac{x^2}{2A}}\,dx = \frac{1}{\sqrt{2\pi A}} \int_{-\infty}^{\infty} \sum_{0}^{\infty} a_n H_n(x) H_m(x)\, e^{-\frac{x^2}{2A}}\,dx$$

$$= a_m A^m m!$$

so $a_n = \frac{1}{A^n n!} \frac{1}{\sqrt{2\pi A}} \int_{-\infty}^{\infty} f(x) H_n(x)\, e^{-\frac{x^2}{2A}}\,dx$

So can exp f(x) in Hermite fns or Hermite polys.

2 1

LD

f_x f

~~12~~
~~9~~
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~~40~~

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~~76~~

6.572 NL no mom eqts 5 May 1961

$52/365$ $\frac{28/3}{23}$

~~28 = Sun 61, Mon 62, Tues 63, Thr 64, Fri 65, 28 Sat 66, 29 Sun, 30 Mon~~

Exp in Hermite polys

$$\frac{1}{\sqrt{2\pi A}}\int_{-\infty}^{\infty} H_m(x) H_n(x) e^{-\frac{x^2}{2A}} dx = A^n n! \, \delta_{mn}$$

$$f(x) = \sum_{n=0}^{\infty} a_n H_n(x)$$

$$\frac{1}{\sqrt{2\pi A}}\int_{-\infty}^{\infty} f(x) H_m(x) e^{-\frac{x^2}{2A}} dx = \frac{1}{\sqrt{2\pi A}}\int_{-\infty}^{\infty} \sum_{n=0}^{\infty} a_n H_n(x) H_m(x) e^{-\frac{x^2}{2A}} dx$$

$$= A^m m! \, a_m$$

$$\Rightarrow a_n = \frac{1}{A^n n! \sqrt{2\pi A}} \int_{-\infty}^{\infty} f(x) H_n(x) e^{-\frac{x^2}{2A}} dx$$

Norm H's

$$\int \frac{H_m(x)}{\sqrt{\sqrt{2\pi A}}\, A^{\frac{m}{2}} \sqrt{m!}} \frac{H_n(x)}{\sqrt{\sqrt{2\pi A}}\, A^{\frac{n}{2}} \sqrt{n!}} e^{-\frac{x^2}{2A}} dx = \delta_{mn}$$

Let $\frac{H_m(x)}{\sqrt{\sqrt{2\pi A}}\, A^{\frac{m}{2}} \sqrt{m!}} = H_m^*(x)$

$$\int_{-\infty}^{\infty} H_m^*(x) H_n^*(x) e^{-\frac{x^2}{2A}} dx = \delta_{mn}$$

$$f(x) = \sum_{0}^{\infty} c_n H_n^*(x)$$

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(2)

Consid. the weighted integral sq error for N terms of H_n^*:

$$\mathcal{E}_{W,N} = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2A}} \left[f(x) - \sum_{n=0}^{N} c_n H_n^*(x)\right]^2 dx$$

$$= \int e^{-\frac{x^2}{2A}} f^2(x)\,dx - 2\sum_{n=0}^{N} c_n \int_{-\infty}^{\infty} f(x) H_n^*(x) e^{-\frac{x^2}{2A}}dx + \sum_n \sum_m c_n c_m \int H_n^*(x) H_m^*(x) e^{-\frac{x^2}{2A}}$$

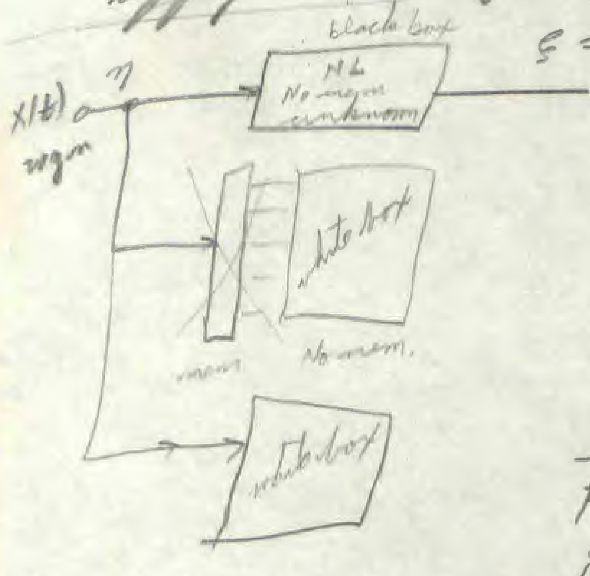
$$= \int e^{-\frac{x^2}{2A}} f^2(x)\,dx - 2\sum_{0}^{N} c_n \int_{-\infty}^{\infty} f(x) H_n^*(x) e^{-\frac{x^2}{2A}}dx + \sum_{0}^{N} c_n^2$$

$$= \int e^{-\frac{x^2}{2A}} f^2(x)\,dx + \sum_{0}^{N} \Big[\underbrace{c_n - \int_{-\infty}^{\infty} f(x) H_n^*(x) e^{-\frac{x^2}{2A}}dx}_{\text{set} = 0.}\Big]^2 - \sum_{0}^{N}\left[\int_{-\infty}^{\infty} f(x) H_n^*(x) e^{-\frac{x^2}{2A}}dx\right]^2$$

Thus if $c_n = \int f(x)\, H_n^*(x) e^{-\frac{x^2}{2A}}dx$, have ~~[illegible]~~ minimized m.s.e. weighted by $e^{-\frac{x^2}{2A}}$

Apply to NL syst w/ no mem;

$\eta = 0 \to$ D.C.



$$\xi = \sum_{n=1}^{\infty} F_n(\eta) = \sum_{1}^{\infty} a_n H_n(\eta) = F(\eta)$$

$$F_1(\eta) = a_1 H_1(\eta) = a_1 \eta$$

$$F_2(\eta) = a_2 H_2(\eta) = a_2(\eta^2 - A)$$

$$F_3(\eta) = a_3 H_3(\eta) = a_3(\eta^3 - 3A\eta)$$

$$\overline{F(\eta) H_m(\eta)} = \sum_{1}^{\infty} a_n \overline{H_n(\eta) H_m(\eta)}$$

$$= \int_{-\infty}^{\infty} \sum_{1}^{\infty} a_n H_n(y) H_m(y) \frac{1}{\sqrt{2\pi A}} e^{-\frac{y^2}{2A}} dy$$

$p_\eta(y)$; $\sigma_\eta^2 = A$

(7)

Note white noise $\Rightarrow A \to \infty$

Consid finite A

{Prev A was variance of each η outp of the mem.
The inp was white, but became colored as passed
through the lin mem syst.

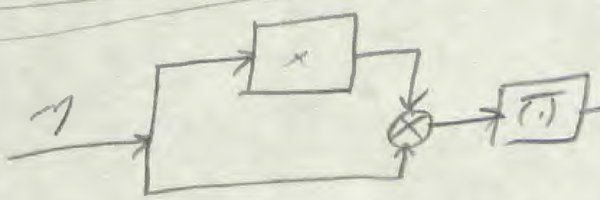
"Bleaching network" to get white noise.

Now back to calculations

$$\overline{F(\eta) H_m(\eta)} = a_m A^m m!$$

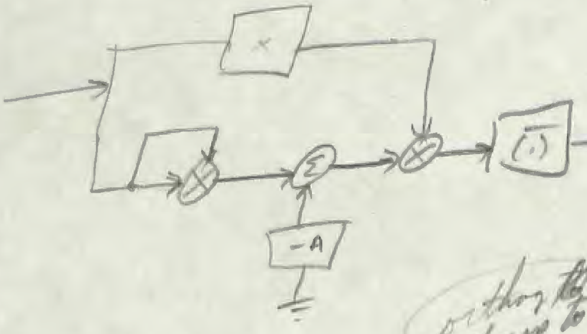
so $$a_m = \frac{\overline{F(\eta) H_m(\eta)}}{A^m m!}$$

criterion is min mse
as weighting fn already
included in prob dist (gaussian)



$$\overline{\xi\eta} = a_1 \overline{\eta^2} = a_1 A$$

$$\Rightarrow a_1 = \frac{\overline{\xi\eta}}{A}$$



$$\overline{\xi(\eta^2 - A)} = a_2 \overline{(\eta^2 - A)^2}$$

$$= a_2 A^2 2!$$

so $$a_2 = \overline{\xi(\eta^2 - A)} \frac{1}{A^2 2!}$$

orthog w/resp to weighting fn

Could use another orthog set; choose fns so that
weighting fn is the prob dist of the input.

8 May 6.572

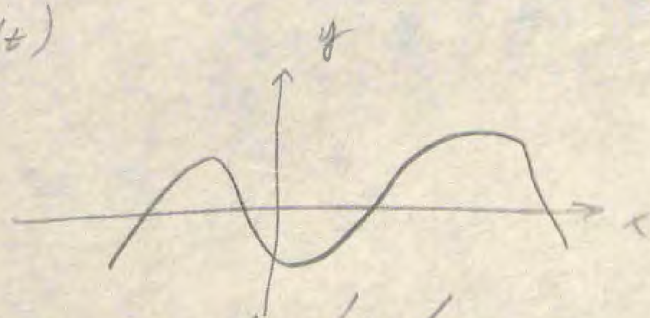
NL NM systs cont

$$\eta \to \xi = F(\eta) = \sum_{1}^{\infty} F_n(\eta) = \sum_{1}^{\infty} a_n H_n(\eta)$$

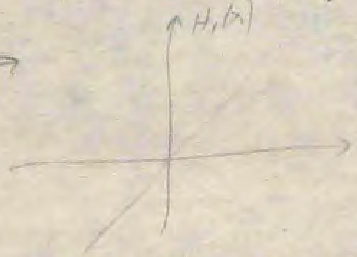
$$x(t) \to y(t) = \sum_{1}^{\infty} a_n H_n[x(t)] = \sum_{1}^{\infty} F_n[x(t)]$$

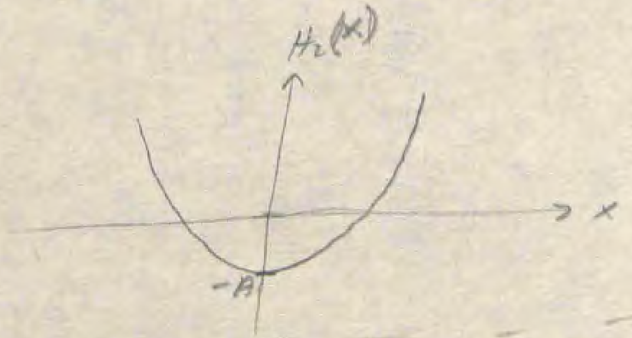
$$F_1[x(t)] = a_1 H_1[x(t)] = a_1 x(t)$$

$$F_2[x(t)] = a_2[x^2(t) - A]$$

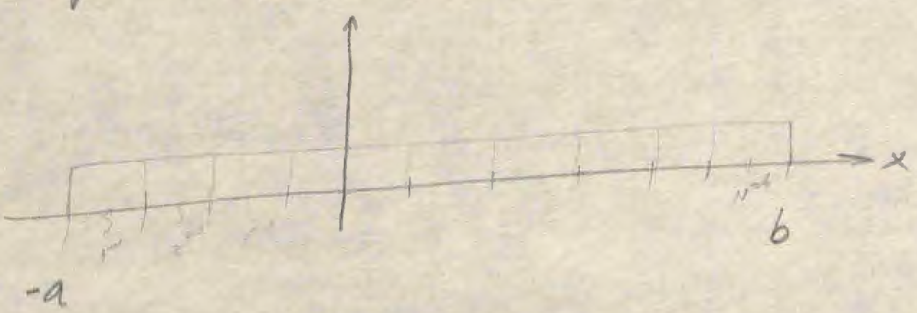


Math we have been using is approx of this fn by the H polys

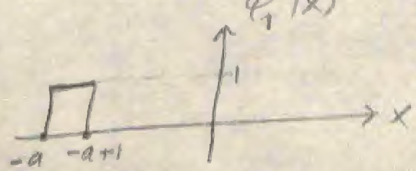




Use of orthog gate fns in NL systs



Use set $\{\varphi_n(x)\}$, $n = 1, \dots, N$



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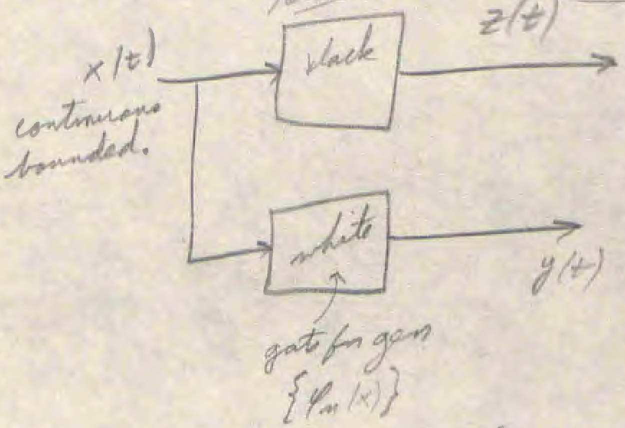
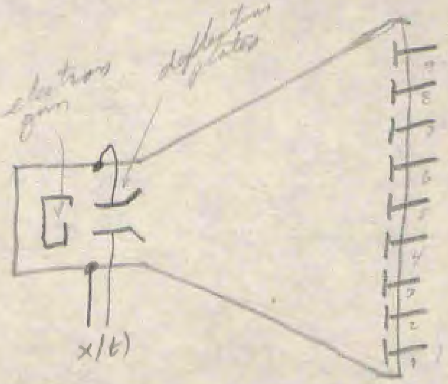
realiz NLNM syst \ or desired output of a NLNM syst



$$y(t) = \sum_{1}^{N} a_n \varphi_n[x(t)],$$

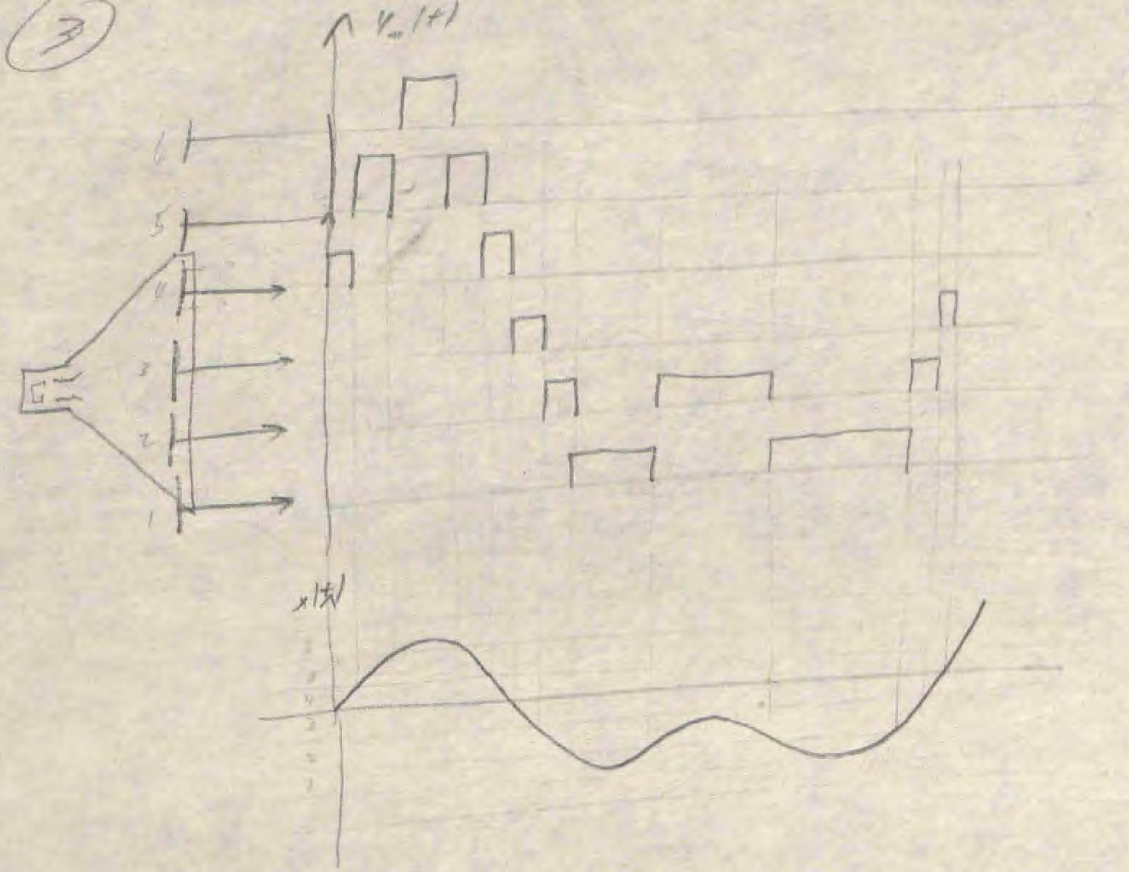
want to pick a's for min mse

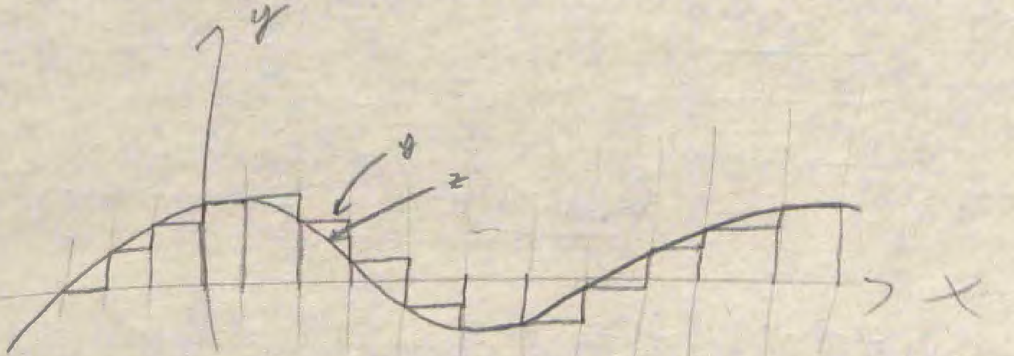


"Level Selector Tube" - Bose
QPR of RLE, Jan '56, July '56

see
"Probability Analyzer" - Burton
QPR of RLE, July '60

(3)





$$\overline{\varphi_m[x(t)]\,\varphi_n[x(t)]} = \overline{\varphi_n^2[x(t)]}\,\delta_{mn} = \overline{\varphi_n[x(t)]}\,\delta_{mn}$$

$$y(t) = \sum_1^N c_n\,\varphi_n^*[x(t)]$$

$$\varphi_n^*[x(t)] = \frac{\varphi_n[x(t)]}{\sqrt{\overline{\varphi_n[x(t)]}}}$$

$$\overline{\varepsilon^2} = \overline{[z(t)-y(t)]^2} = \overline{\left[z(t) - \sum_1^N c_n\,\varphi_n^*[x(t)]\right]^2}$$

(4)

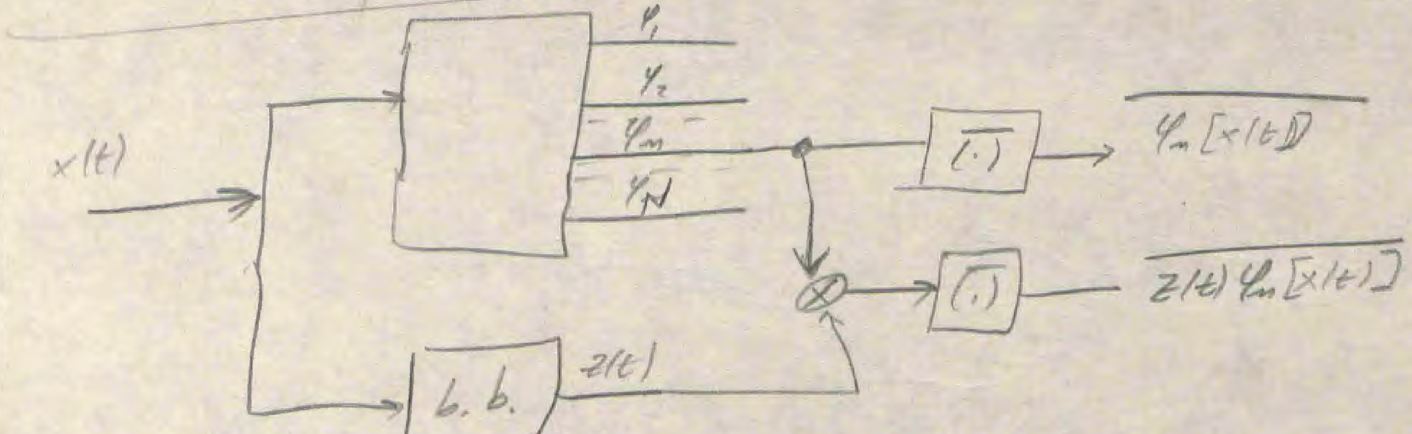
$$\overline{\varepsilon^2} = \overline{z^2(t)} + \sum_{1}^{N} \left[c_n - \overline{z(t)\,\varphi_n^*[x(t)]} \right]^2 - \sum_{1}^{N} \overline{z(t)\,\varphi_n^*[x(t)]}^{\,2}$$

For min $\overline{\varepsilon^2}$, $\quad c_n = \overline{z(t)\,\varphi_n^*[x(t)]}$

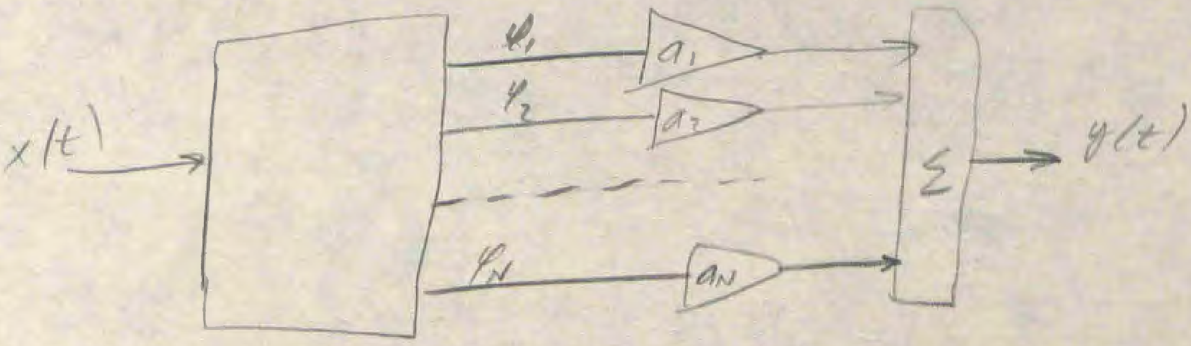
for min mse

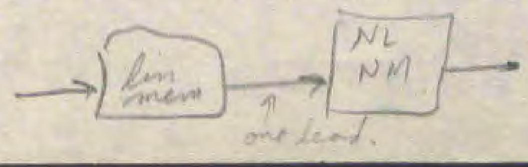
$$y(t) = \sum_{1}^{N} \frac{c_n}{\sqrt{\overline{\varphi_n[x(t)]}}}\,\varphi_n[x(t)] = \sum_{1}^{N} a_n\,\varphi_n[x(t)]$$

$$a_n = \frac{\overline{z(t)\,\varphi_n[x(t)]}}{\sqrt{\overline{\varphi_n[x(t)]}}}$$



Synthesized ckt is:





12 May

6.572 GatE fuNctI°nS

8

$2 \times 137 \times 10^{256} \approx 3 \times 10^{258}$

M LWS
LS CWJ
C C C
J J
CJW

$$\overline{\varepsilon^2_{w,N}} = \overline{\left[z(t) - \sum_1^N a_m \varphi_m[x(t)] \right]^2 w^2(t)}$$

$w^2(t)$ = positive weighting fn.

E.g., supp

$$\overline{\varepsilon^2_{w,N}} = \overline{\left\{ z(t)w(t) - \sum_1^N a_m \varphi_m[x(t)]\, w(t) \right\}^2}$$

Determine orthonormal set $\{ \varphi^*_{m(w)}[x(t)] \}$

from the set $\{ \varphi_m[x(t)]\, w(t) \}$

so $\overline{\varphi^*_{n(w)}[x(t)]\, \varphi^*_{m(w)}[x(t)]} = \delta_{mn}$

Now $\overline{\varphi_n[x(t)]\, w(t)\, \varphi_m[x(t)]\, w(t)} = \overline{\varphi_m[x(t)]\, \varphi_n[x(t)]\, w^2(t)} = \overline{\varphi_m[x(t)]\, w^2(t)}\, \delta_{mn}$

$= \overline{\varphi^2_m[x(t)]\, w^2(t)}\, \delta_{mn} =$

so $\varphi^*_{m(w)}[x(t)] = \dfrac{\varphi_n[x(t)]\, w(t)}{\left\{ \overline{\varphi_m[x(t)]\, w^2(t)} \right\}^{1/2}}$

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(2)

$$\overline{\mathcal{E}^2_{w,N}} = \overline{\left\{ z(t)\,w(t) - \sum_1^N b_n\, \varphi^*_{n(w)}\left[x(t)\right] \right\}^2}$$

$$= \overline{z^2(t)\,w^2(t)} - 2\sum_1^N b_n\, \overline{z(t)\,w(t)\,\varphi^*_{n(w)}\left[x(t)\right]} + \sum_1^N b_n^2$$

$$= \overline{z^2(t)\,w^2(t)} + \sum_1^N \left[b_n - \overline{z(t)\,w(t)\,\varphi_{n(w)}\left[x(t)\right]} \right]^2 - \sum_1^N \overline{z(t)\,w(t)\,\varphi^*_{n(w)}\left[x(t)\right]}^{\,2}$$

To $\min\limits_{b_n} \overline{\mathcal{E}^2_{w,N}}$, must have $b_n = \overline{z(t)\,w(t)\,\varphi^*_{n(w)}\left[x(t)\right]}$

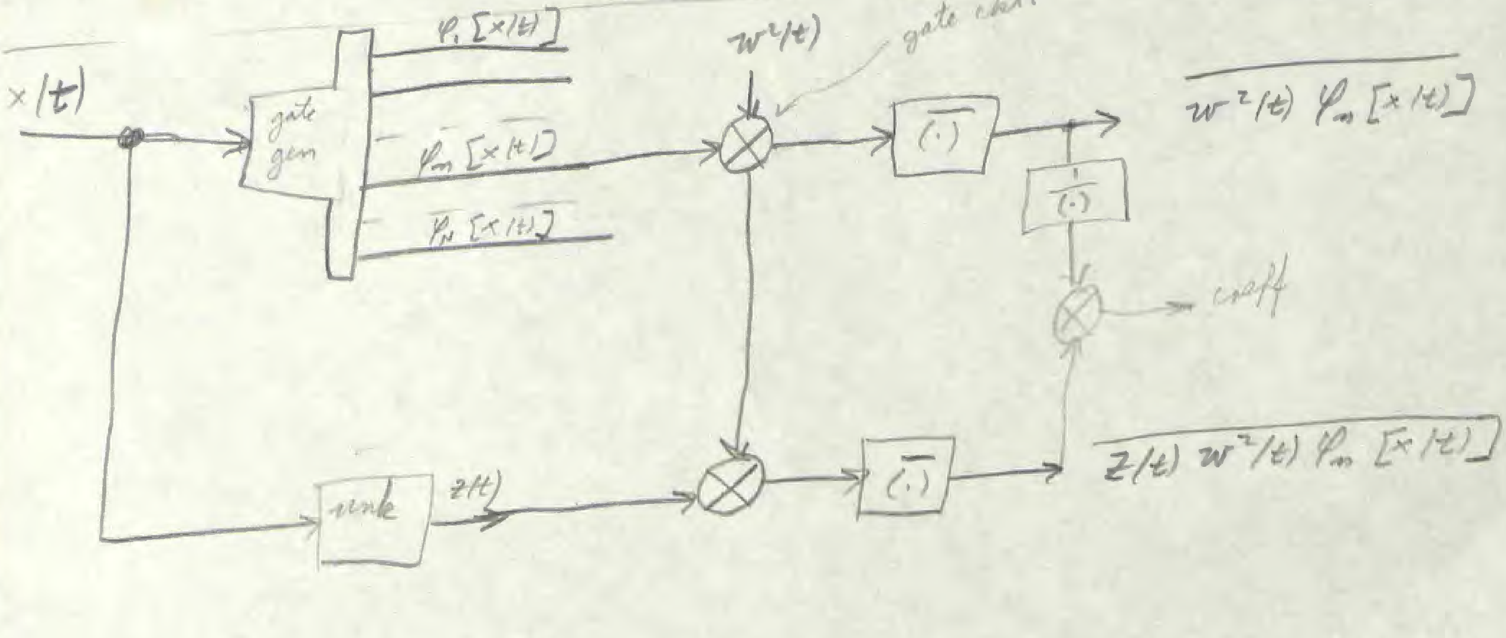
$$b_n = \frac{\overline{z(t)\,w(t)\,\varphi_n\left[x(t)\right]w(t)}}{\sqrt{\overline{\varphi_n\left[x(t)\right]w^2(t)}}} =$$

now

$$\sum_1^N a_n\,\varphi_n\left[x(t)\right]w(t) = \sum_1^N b_n\,\varphi^*_{n(w)}\left[x(t)\right] = \sum_1^N \frac{\overline{z(t)\,w^2(t)\,\varphi_n\left[x(t)\right]}}{\sqrt{\overline{\varphi_n\left[x(t)\right]w^2(t)}}} \left[\frac{\varphi_n\left[x(t)\right]}{\sqrt{\overline{\varphi_n\left[x(t)\right]w^2(t)}}}\right]$$

or

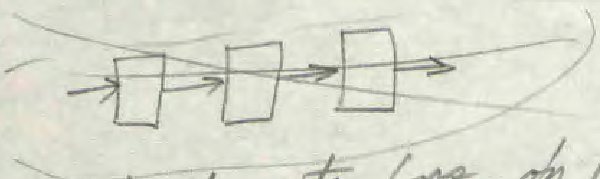
$$a_n = \frac{\overline{w^2(t)\,z(t)\,\varphi_n\left[x(t)\right]}}{\overline{w^2(t)\,\varphi_n\left[x(t)\right]}}$$



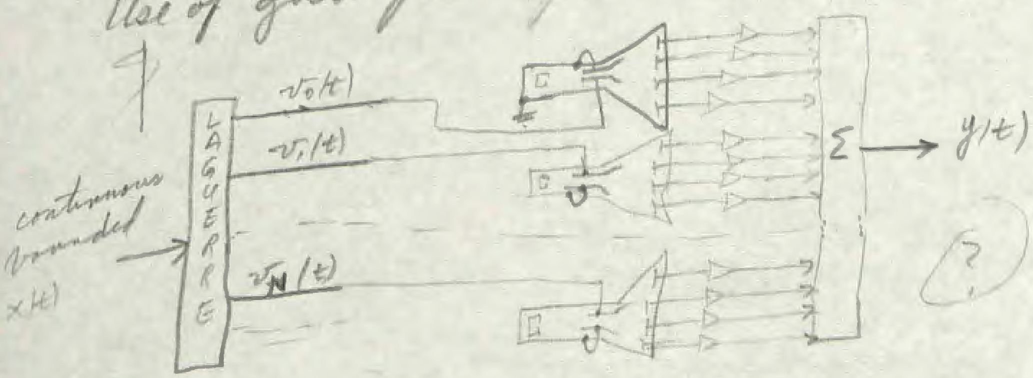
3 bjorcsh

~~supp~~ $\cancel{w^2(t) = \frac{1}{|x(t)-z(t)|^2}}$

e.g. $w^2(t) = [x(t)-z(t)]^2$ or $\frac{1}{|x(t)-z(t)|^2}$



Use of gate fns on NL w/ Mem systs.



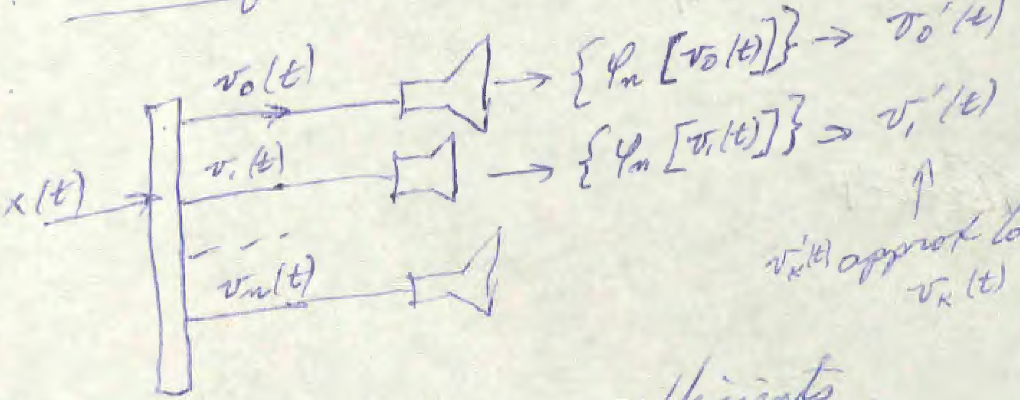
Mathematically,

$$y(t) = \sum_{m_0=1}^{M} \sum_{m_1=1}^{M} \cdots \sum_{m_N=1}^{M} a_{m_0 m_1 \ldots m_N} \varphi_{m_0}[v_0(t)]\,\varphi_{m_1}[v_1(t)] \cdots \varphi_{m_N}[v_N(t)].$$

15 May 65 72

~~Eqnts~~

Gate fns (incl mem):



$m+1 = M = \#$ of Laguerre coefficients.

Treat each Laguerre outp as a simple time fn going into the non-lin F fn.

$$y(t) = F[v_0'(t), v_1'(t), \ldots, v_n'(t)]$$

$$= \sum_{i=0}^{n} c_i v_i'(t) + \sum_{i=0}^{m}\sum_{j=0}^{m} c_{ij} v_i'(t) v_j'(t) + \ldots$$

Look at linear terms:

$$c_0 v_0'(t) = c_0 \sum_{m_0=1}^{N} b_{m_0} \varphi_{m_0}[v_0(t)] = \sum_{m_0=1}^{N} A_{m_0} \varphi_{m_0}[v_0(t)]$$

$$c_1 v_1'(t) = c_1 \sum_{m_1=1}^{N} b_{m_1} \varphi_{m_1}[v_1(t)] = \sum_{m_1=1}^{N} A_{m_1} \varphi_{m_1}[v_1(t)]$$

2nd order terms:

$$\cancel{c_{00} v_0'(t) v_0'(t) = \sum_{m_0=1}^{N} A_{m_0 m_0} \varphi^2_{m_0}[v_0(t)]}$$

$$c_{00} v_0'(t) v_0'(t) = \sum_{m_0=1}^{N}\sum_{n_0=1}^{N} A_{m_0 n_0} \varphi_{m_0}[v_0(t)] \varphi_{n_0}[v_0(t)]$$

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2

$$\cancel{c_{00}\, v_0'(t)\, v_0'(t) =}$$

$$c_{01}\, v_0'(t)\, v_1'(t) = \sum \sum A_{m_0 m_1} \varphi_{m_0}[v_0(t)] \varphi_{m_1}[v_1(t)]$$

Now $\sum_{m_0=1}^{N} \varphi_{m_0}[v_0(t)] = 1$

$$\sum_{m_1=1}^{N} \varphi_{m_1}[v_1(t)] = 1$$

$$c_0 v_0'(t) = \sum_{m_0=1}^{N} A_{m_0} \varphi_{m_0}[v_0(t)] \sum_{m_1=1}^{N} \sum_{m_2=1}^{N} \cdots \sum_{m_n=1}^{N} \varphi_{m_1}[v_1(t)] \cdots \varphi_{m_n}[v_n(t)]$$

$$= \sum_{m_0=1}^{N} \sum \cdots \sum_{m_n=1}^{N} A_{m_0} \varphi_{m_0}[v_0(t)] \cdots \varphi_{m_n}[v_n(t)]$$

Sim $c_1 v_1'(t) = \sum \sum \cdots \sum A_{m_1} \varphi_{m_0}[v_0(t)] \cdots \varphi_{m_n}[v_n(t)]$

Sim $c_{01}\, v_0'(t)\, v_1'(t) = \sum \sum A_{m_0 m_1} \varphi_{m_0}[v_0] \varphi_{m_1}[v_1]$

$$= \sum_{m_0} \cdots \sum_{m_n} A_{m_0 m_1} \varphi_{m_0}(v_0) \cdots \varphi_{m_n}(v_n)$$

so $\sum c_i v_i(t) = \sum_{m_0} \cdots \sum_{m_n} \sum_i A_{m_i} \varphi_{m_0}(v_0) \cdots \varphi_{m_n}(v_n)$

$$\varphi_{m_0}[v_0(t)] \cdots \varphi_{m_n}[v_n(t)]$$

$$y(t) = \sum_{m_0} \cdots \sum_{m_n} a_{m_0 \ldots m_n} \varphi_{m_0}[v_0(t)] \cdots \varphi_{m_n}[v_n(t)]$$

$a_{m_0 \ldots m_n} = \sum_i A_{m_i} + \sum$?

$\{\varphi_{m_0}[v_0(t)] \cdots \varphi_{m_n}[v_n(t)]\}$ is an orthog set.

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Finding the a's:

$$y(t) = \sum_{m_0} \cdots \sum_{m_n} a_{m_0 \cdots m_n} \varphi_{m_1}[v_1(t)] \cdots \varphi_{m_n}[v_n(t)]$$

weighted mse is

$$\overline{\mathcal{E}^2_{w,N,M}} = \overline{\left[z(t) - \sum_{m_0} \cdots \sum_{m_n} a_{m_0 \cdots m_n} \varphi_{m_0}[v_0(t)] \cdots \varphi_{m_n}[v_n(t)]\right]^2 w^2(t)}$$

$\boxed{N = \#\text{ of levels of tubes}}$

def weighted orthonormal set

$$\left\{ \left(\varphi_{m_0}[v_0(t)] \cdots \varphi_{m_n}[v_n(t)]\right)^*_w \right\}$$

where $\left(\varphi_{m_0}[v_0(t)] \cdots \varphi_{m_n}[v_n(t)]\right)^*_w = \dfrac{\varphi_{m_0}[v_0(t)] \cdots \varphi_{m_n}[v_n(t)]\, w(t)}{\sqrt{\overline{\varphi_{m_0}[v_0(t)] \cdots \varphi_{m_n}[v_n(t)]\, w^2(t)}}}$

$$\overline{\left(\varphi_{m_0}[v_0(t)] \cdots \varphi_{m_n}[v_n(t)]\right)^*_w \left(\varphi_{k_0}[v_0(t)] \cdots \varphi_{k_n}[v_n(t)]\right)^*_w} = \begin{cases} 1 & \text{if } m_i = k_i \text{ all } i = 0, \ldots, n \\ 0 & \text{otherwise} \end{cases}$$

$$\overline{\mathcal{E}^2_{w,N,M}} = \overline{\left[z(t)w(t) - \sum \cdots \sum b_{m_0 \cdots m_n} \left(\varphi_{m_0}[v_0] \cdots \varphi_{m_n}(v_n)\right)^*_w\right]^2}$$

for min $\overline{\mathcal{E}^2_{w,N,M}}$,

$$b_{m_0 \cdots m_n} = \frac{\overline{w^2(t)\, z(t)\, \varphi_{m_0}[v_0(t)] \cdots \varphi_{m_n}[v_n(t)]}}{\sqrt{\overline{w^2(t)\, \varphi_{m_0}[v_0(t)] \cdots \varphi_{m_n}[v_n(t)]}}}$$

$$a_{m_0 \cdots m_n} = \frac{\overline{w^2(t)\, z(t)\, \varphi_{m_0}[v_0(t)] \cdots \varphi_{m_n}[v_n(t)]}}{\overline{w^2(t)\, \varphi_{m_0}[v_0(t)] \cdots \varphi_{m_n}[v_n(t)]}}$$

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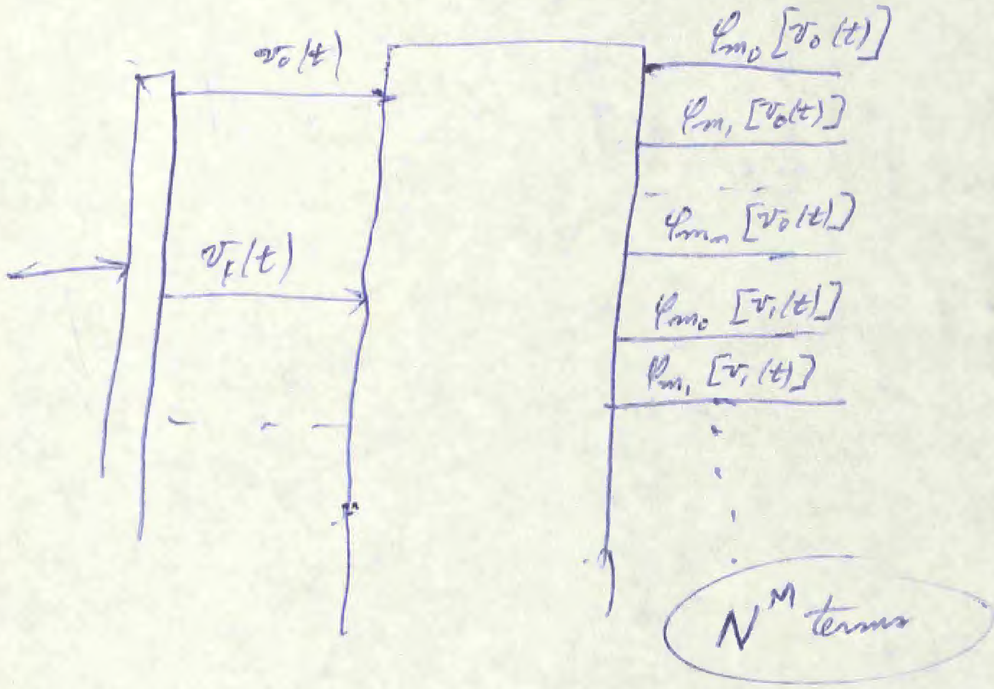
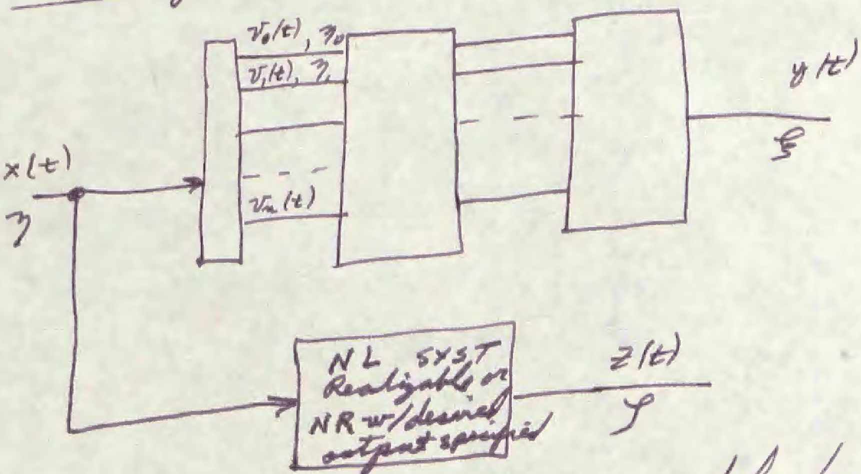

$v_0(t)$
$v_1(t)$
$\varphi_{m_0}[v_0(t)]$
$\varphi_{m_1}[v_0(t)]$
$\varphi_{m_n}[v_0(t)]$
$\varphi_{m_0}[v_1(t)]$
$\varphi_{m_1}[v_1(t)]$
N^M terms

P_{xyz} ~~$\mathcal{H}$~~ $P(x)$, $P(x,y)$, $P(x,y,z)$

17 May 6.572 :



time var.	ensemb	range
$x(t)$	η	
$y(t)$	ξ	r
$z(t)$	ζ	s
$v_0(t)$	η_0	u_0
$v_i(t)$	η_i	u_i
$v_n(t)$	η_n	u_n

What is mse on ensemble basis?

$\overline{\epsilon^2} = \overline{(\zeta - \xi)^2}$ which is

Need joint p. density of $\zeta, \eta_0, \ldots, \eta_n$

$P_{\zeta, \eta_0, \ldots, \eta_n}(s, u_0, \ldots, u_n)$

so $\overline{\epsilon^2} = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (s-r)^2 P_{\zeta, \eta_0, \ldots, \eta_n}(s, u_0, \ldots, u_n)\, ds\, du_0\, du_1 \ldots du_n$

What is relationship between ξ η for min mse?

Introduce calc of variation, variables:

a = parameter independent of $\eta_0, \ldots, \eta_n$

ρ = variation in ξ

ρ is a fn of $(\eta_0, \ldots, \eta_n)$

q = range variable for ρ

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2

$$I = \overline{\epsilon^2} + \delta\overline{\epsilon^2} = \int\cdots\int [s-(r+aq)]^2 P_{\xi,\eta_0,\ldots,\eta_m}(s,u_0,\ldots,u_m)\,ds\,du_0\cdots du_m$$

$$\boxed{\begin{gathered} p(s|u) = p(s|x)\,p(x|u) \\ p(s|u) = p(s|x)\,p(x,u) \end{gathered}}$$

$$\left.\frac{\partial I}{\partial a}\right|_{a=0} = 0 \text{ for all } p \Rightarrow \min \overline{\epsilon^2}$$

$$\frac{\partial I}{\partial a} = -2\int\cdots\int [s-(r+aq)]\,q\,P_{\xi,\eta_0,\ldots,\eta_m}(s,u_0,\ldots,u_m)\,ds\,du_0\cdots du_m$$

$$0 = \left.\frac{\partial I}{\partial a}\right|_{a=0} \Rightarrow 0 = \int\cdots\int (s-r)\,q\,P_{\xi,\eta_0,\ldots,\eta_m}(s,u_0,\ldots,u_m)\,ds\,du_0\cdots du_m$$

for all p

or $$0 = \int\cdots\int (s-r)\,q\,P_{\eta_0,\ldots,\eta_m}(u_0,\ldots,u_m)\,P_{\xi|\eta_0,\ldots,\eta_m}(s|u_0,\ldots,u_m)\,ds\,du_0\cdots du_m \quad \text{all } p$$

or $$0 = \int\cdots\int q\,\underbrace{P_{\eta_0\ldots\eta_m}(u_0,\ldots,u_m)}_{\Leftrightarrow\, p(\eta_0,\ldots,\eta_m)}\,du_0\cdots du_m \int (s-r)\,P_{\xi|\eta_0,\ldots,\eta_m}(s|u_0,\ldots,u_m)\,ds \quad \text{all } p$$

Now this must hold for all q (or p), so

$$\int (s-r)\,P_{\xi|\eta_0,\ldots,\eta_m}(s|u_0,\ldots,u_m)\,ds = 0$$

or ~~$\int s P_{\xi|\eta_0}$~~ or $\int s\,P_{\xi|\eta_0,\ldots,\eta_m}(s|u_0,\ldots,u_m)\,ds = \int r\,P_{\xi|\eta_0\ldots\eta_m}(s|u_0,\ldots,u_m)\,ds$

r is indep of s, so $\qquad = r$

not restricted to Gauss inp.

$$\boxed{\int s\,P_{\xi|\eta_0,\ldots,\eta_m}(s|u_0,\ldots,u_m)\,ds = r \quad \text{is condx for min mse}}$$

gives condx but no procedure for building a model system.

18 May 6.572 :

$$\int\cdots\int (s-r)\, P_{\underline{s},\underline{u}}(s,\underline{u})\, ds\, d\underline{u}$$
$$= \int\cdots\int (s-r)\, P_{s|\underline{u}}(s|\underline{u})\, P_{\underline{u}}(\underline{u})\, ds\, d\underline{u}$$
$$= \int\cdots\int (s-r)\, P_{s|\underline{u}}(s|\underline{u})\, ds \int P_{\underline{u}}(\underline{u})\, d\underline{u}$$

$r = r(\underline{u}) \qquad s = s(\underline{u})$

Diff eqs & final representation for a NL syst

Refs: "Synthesis of Optimum NL Control Systems" ScD Thesis EE Van Trees, June 1961

Theses by Barrett & D. George ref at beginning of course.

$$P_1(y, y', y'', \ldots, y^{(r)}) = P_2(x, x', \ldots, x^{(s)})$$ polys in derivs.

e.g. by Barrett

$$y'' + a_1 y' + a_2 y + a_3 y^3 = x$$

go to final representation

Let $D = \frac{d}{dt}(\cdot) = (\cdot)'$

$$(D^2 + a_1 D + a_2)\, y + a_3 y^3 = x$$

$$C = D^2 + a_1 D + a_2$$

$$Cy + a_3 y^3 = x$$

Expand y in power series $y = A_1 x + A_2 x^2 + \cdots$
$= y_1 + y_2 + \cdots$

$$C\left[A_1 x + A_2 x^2 + \cdots\right] + a_3\left[A_1 x_1 + A_2 x^2 + \cdots\right]^3 = x$$

(2)

$$C(A_1 x + A_2 x^2 + \dots) + a_3\left[A_1^3 x^3 + 3A_1^2 A_2 x^4 + (3A_1 A_2^2 + A_1^2 A_3)x^5 + \dots\right]$$

$CA_1 = 1$

$CA_2 = 0$

$CA_3 + a_3 A_1^3 = 0$

$CA_4 + 3a_3 A_1^2 A_2 = 0$

etc

$CA_1 = 1 \Rightarrow A_1 C x = x$ or $A_1 = C^{-1}$

e.g. $Cy = x$, $y = C^{-1}x$

$(D^2 + a_1 D + a_2)\, y = x$

sol known $y = \int_{-\infty}^{\infty} h_1(\tau_1)\, x(t-\tau_1)\, d\tau_1$

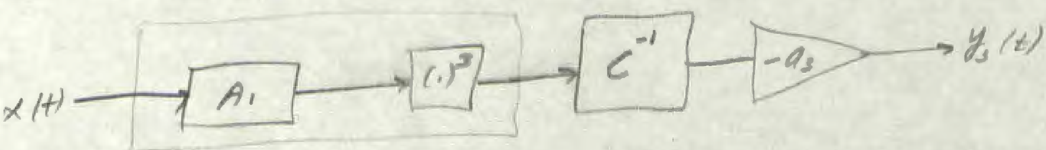
h_1 is transform of $\dfrac{1}{(jw)^2 + a_1 jw + a_2}$

or $y_1(t) = \int h_1(\tau_1)\, x(t-\tau_1)\, d\tau_1$

$CA_2 = 0 \Rightarrow A_2 = 0$

$CA_3 + a_3 A_1^3 = 0 \Rightarrow A_3 = -a_3 C^{-1} A_1^3$

$A_3 x^3 = -a_3 C^{-1}[A_1 x]^3$



$$[A_1 x(t)]^3 = \left[\int h_1(\tau_1)\, x(t-\tau_1)\, d\tau_1\right]^3 = \iiint h_1(\tau_1) h_1(\tau_2) h_1(\tau_3)\, x(t-\tau_1)\, x(t-\tau_2)\, x(t-\tau_3)\, d\tau_{123}$$

$$C^{-1}[A_1 x(t)]^3 = \int h_1(\tau_4)\, d\tau_4 \iiint h_1(\tau_1) \dots h_1(\tau_3)\, x(t-\tau_1-\tau_4)\, x(t-\tau_2-\tau_4)\, x(t-\tau_3-\tau_4)\, d\tau_{123}$$

$$= \int \dots \int h_1(\tau_1) \dots h_1(\tau_4)\, x(t-\tau_1-\tau_4) \dots d\tau_{1234}$$

(3)

Let $\tau_5 = \tau_1 + \tau_4$, $\tau_6 = \tau_2 + \tau_4$, $\tau_7 = \tau_3 + \tau_4$ so $\begin{cases} \tau_1 = \tau_5 - \tau_4 \\ \tau_2 = \\ \tau_3 = \end{cases}$

$$C^{-1}[\ \ ]^3 = \int \cdots \int h_1(\tau_5 - \tau_4)\, h_1(\tau_6 - \tau_4)\, h_1(\tau_7 - \tau_4)\, h_1(\tau_4)\, x(t-\tau_5)\, x(t-\tau_6)\, x(t-\tau_7)\, d\tau_{567}$$

Let $h_3(\tau_5, \tau_6, \tau_7) = \int h_1(\tau_5 - \tau_4)\, h_1(\tau_6 - \tau_4)\, h_1(\tau_7 - \tau_4)\, h_1(\tau_4)\, d\tau_4$

$$A_1 x^3 = \int h_1(\tau_1)\, x^3(t-\tau_1)\, d\tau_1$$

$$A_1^2 x^3 = \int h_1(\tau_2)\, d\tau_2 \int h_1(\tau_1)\, x^3(t - \tau_2 - \tau_1)\, d\tau_1$$

$$= \int h_1(\tau_1)\, h_1(\tau_2)\, x^3(t - \tau_2 - \tau_1)\, d\tau_1\, d\tau_2$$

$$= \int h_1(\tau_1)\, h_1(\tau_3 - \tau_1)\, x^3(t - \tau_3)\, d\tau_1\, d\tau_3$$

?

$$C^{-1}\left[\boxed{A_1}\, x(t)\right]^3 = \iiint h_3(\tau_1 \tau_2 \tau_3)\, x(t-\tau_1) \cdots d\tau_{123}$$

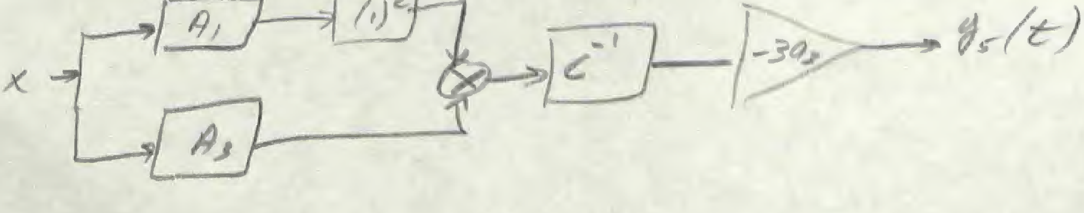
$$y_3(t) = -a^3 \iiint - - \ - \ -$$

$$C A_4 + 3a_3 A_1^2 A_2 = 0$$

$A_2 = 0$ so $C A_4 = 0$ or $\underline{A_4 = 0}$

$$C A_5 + 3a_3 A_1^2 A_3 = 0$$

$$A_5 = -3a_3 C^{-1} A_1^2 A_3$$



Let $\quad Q^{(3)}_{ijk}(\underline{\gamma}) =$