6.536

SYSTEMS ENGINEERING
AND
OPERATIONS RESEARCH

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SYSTEMS ENGINEERING + OPERATIONS RESEARCH

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The course will deal with complex man-machine systems from two viewpoints:

- Analysis (operations research)
- Synthesis (systems engineering)

Subject matter includes:

TERM PROJECT

Probability
Optimization
Control
Information

Markov processes
dynamic programming
simulation - representation of complex systems on computers

[A large part of the problems is to decide on an overall goal and stick to it.]

Organized operations research groups:

Army : ORO
Navy : OEG
Air Force : RAND [also Chief of Op. Analysis]

Private industrial groups
Consulting groups
ADL; Case; SRI.

Definition of a System

input → system → output

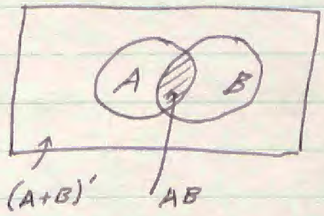
(1) logical operation
(2) physical equipment
(3) purpose to perform
(4) large + complex

(5) some degree of automaticity
(6) probablistic elements
(7) competitive forces

E.g.: MTA, Bell System, etc.

Algebra of Events

A means that the event "A" has occured
A' " " " " " " not occured and is called the complement of A.
$A+B$ means either A or B or both have occured
AB means both A and B occur.



$$[f(+,\cdot,A,B)]' = f(\cdot,+,A',B')$$

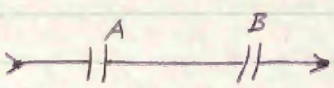
Relations:

$$A + AB = A \qquad \underline{A + A'B = A + B}$$
$$A(A+B) = A$$

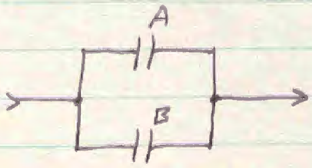
$$\left.\begin{array}{ll} AA' = 0 & ;\ \text{never occurs} \\ A + A' = I & ;\ \text{always occurs.} \end{array}\right\} \quad \begin{array}{l} AI = A \\ A0 = 0. \end{array}$$

Switching circuits:

$A \to A$ closed (conducting)
$A' \to A$ open (not conducting)

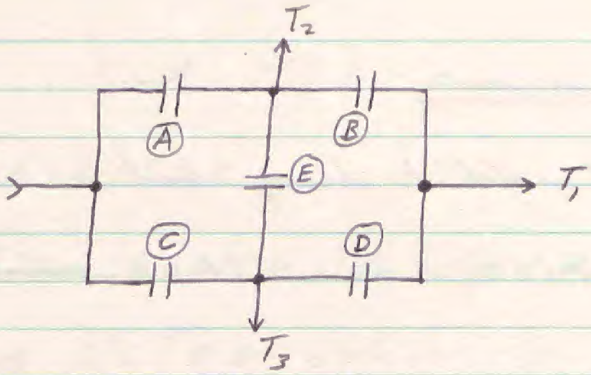


$T = AB$



$T = A + B$

<u>Example</u>:



This might represent ~~of~~ a communication network ~~to~~ linking LA, Chicago, Dallas, & NYC. We can assign a probability to each link that it will be operating. Then we want to find the conditions &/or probability that a given set of networks is or is not operating.

For example, we may want to know under what conditions will $T_1 \neq 0$ and $T_2 = T_3 = 0 = T_2 T_3$. We then form the transmission expression for $T_1 T_2' T_3'$.

$$T_1 = A(B+DE) + C(D+BE)\ ; \quad T_1' = [A' + B'(D'+E')][C' + D'(B'+E')]$$

$$T_2 = A + C(E+BD)\ ; \quad T_2' = A'[C' + E'(B'+D')]$$

$$T_3 = C + A(E+BD)\ ; \quad T_3' = C'[A' + E'(B'+D')]$$

Now,

$T_1 T_2' T_3' = 0$ since $AA' = CC' = 0$ & also <u>by inspection</u>

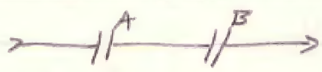
$$T_1 T_2' T_3 = A'B'CDE'$$

$$= [\cancel{A}(\cancel{B}+\cancel{D}E) + C(D+BE)]\, A'\, [\cancel{C'} + E'(B'+D')][C + \cancel{A}(\cancel{E}+\cancel{BD})]$$

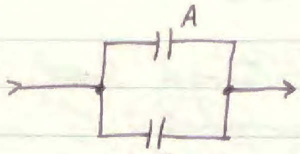
$$= A'C[E'(B'+D')(D+\cancel{BE})] = A'CE'(B'+D')D = A'B'CDE'$$

Probabilistic switching circuits

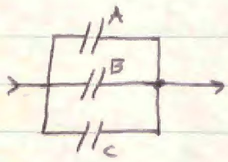
Suppose the probability of switch A being closed is $P(A) = a$ etc., then:



$P(AB) = ab$ (A & B ~~mutually~~ independent)



$P(A+B) = a + b - ab$



$P(A+B+C) = a+b+c-ab-ac-bc+abc$

To simplify the finding of the expressions for the probability of transmission, we use the algebra of events: Find T as a combination of mutually exclusive events; then the probabilities can be written down directly.

E.G. : $T = A+B+C = A + A'(B+C)$

$= A + A'(B + B'C)$ *

$P(T) = a + (1-a)(b + [1-b]c)$

OR $T = A+B+C = (A'B'C')'$

$P(T) = 1 - (1-a)(1-b)(1-c)$

Let $\oplus$ denote that $A \oplus B$ means ~~that~~ $A+B$ when A & B are mutually exclusive events.

<u>Examples:</u> Find $P(T_2)$

$$T_2 = A + CE + BCD$$

$$= A + C(E+BD) = A \oplus A'C(E+BD)$$

$$= A \oplus A'C(E \oplus E'BD)$$

Then $P(T_2) = a + (1-a)c\left[e + (1-e)bd\right]$

Find $P(T_1)$:

$$T_1 = A(B+DE) + C(D+BE)$$

$$= AB + CD + (AD+BC)E$$

$$= AB \oplus (A'+B')\left[CD + (AD+BC)E\right]$$

$$= AB \oplus (A'+B')\left[CD \oplus (C'+D')(AD+BC)E\right]$$

$$= AB \oplus (A'+B')\left[CD \oplus (AC'D + BCD')E\right]$$

$$= AB \oplus (A'+B')CD \oplus (AB'C'D \oplus A'BCD')E$$

So $P(T_1) = ab + (1-ab)cd + e\left[a(1-b)(1-c)d + (1-a)bc(1-d)\right]$

Sample Space :

To find the probability of any event, we graphically display every possible event. The probability of any event (A) is then just the sum of the probabilities of all events satisfying the event (A).

<u>Example</u> : Throw a pair of dice twice giving two 2-digit numbers. Find the probability that the same number appears both times.

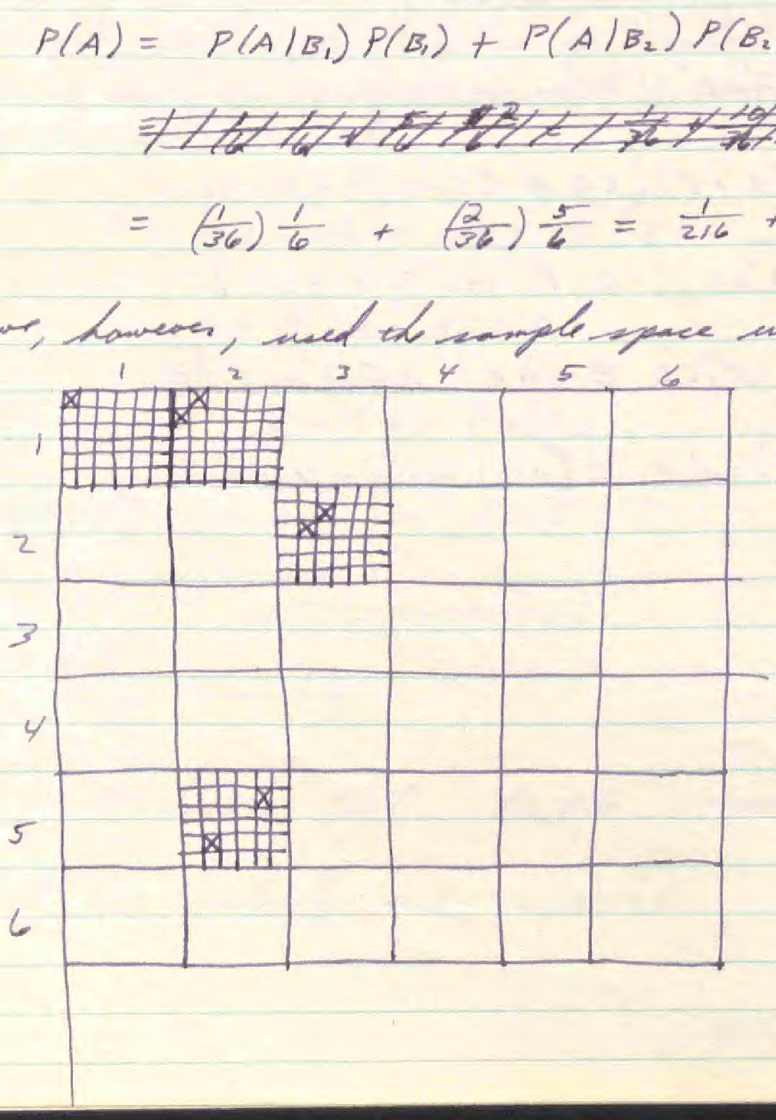
We can do this problem without the sample space; consider two mutually exclusive outcomes

B_1 : first toss is a pair
B_2 : " " " not a pair.

Now, $P(A) = P(A|B_1)\,P(B_1) + P(A|B_2)\,P(B_2)$

$$= \left(\frac{1}{36}\right)\frac{1}{6} + \left(\frac{2}{36}\right)\frac{5}{6} = \frac{1}{216} + \frac{10}{216} = \frac{11}{216}.$$

We have, however, used the sample space unconsciously at least:



Each large square represents a possible outcome of the first throw of the dice. The 36 smaller squares in each large square represent a possible outcome of the second throw. Since each small square is an equally likely event, the probability of A is just the sum of all events satisfying A divided by the total number of squares (1296). If a pair is thrown, there is only one outcome satisfying A; viz., the same pair is thrown on the second throw. At any pt. off the diagonal, the numbers can be formed two different ways, so we have

$$P(A) = \frac{1}{1296}\left[6 + 30(2)\right] = \frac{1}{1296}\left[66\right] = \frac{11}{216}$$

We can also find joint probabilities by this device:

$$P(A|B_1) = \frac{P(AB_1)}{P(B_1)} = \frac{6/36^2}{1/6} = \frac{1}{36}$$

$$= \frac{1}{6}\left(\frac{1}{6}\right) = \text{a specific portion of a restricted sample space.}$$

We can find just about any condition by counting up the small boxes satisfying the condition.

E.g.: the sum of all 4 digits is > 12.
: the sum of the digits on the first throw is even + on the second, odd.

Expected value for discrete random variable:

$$E(x) = \frac{\sum x_i P(x_i)}{\sum P_i}$$

In the event of restricted probabilities, we must normalize to the event.

E.g.: $E(\text{product}) = \frac{1}{36}(1+2+3+\cdots+36) = \frac{441}{36} = 12.2$

$E(\text{product} \mid \text{both \#'s} < 4) = \dfrac{(36)\frac{1}{36}}{9\,\frac{1}{36}} = 4.$

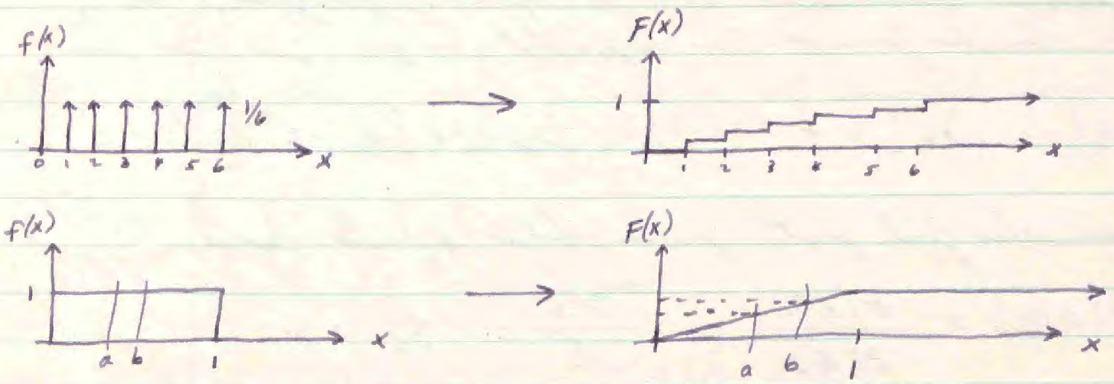
for one throw of 2 dice.

Continuous probability functions :

Density function : $f(x) = \frac{d}{dx} P(x)$

Cumulative distribution : $F(x) = \int_{-\infty}^{x} f(x)\,dx = P(x \le X)$

$$P(a < x < b) = \int_a^b f(x)\,dx = F(b) - F(a)$$



We can have ~~several~~ multi-dimensional functions as well.

Now we can extend our idea of the sample space to include continuous random variables; we have done so above for the one-dimensional case by saying $P(a<x<b) = \int_a^b f(x)\,dx$.

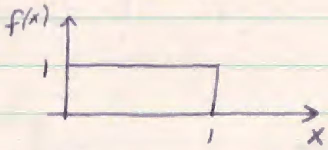
For two (& analogously for more than 2) dimensions, we extend this idea to

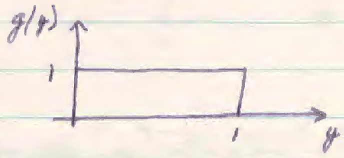
$$P(x \text{ in } S) = \iint_S h(x,y)\,dx\,dy.$$

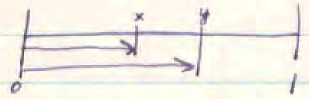
That is, we just integrate $h(x,y)$ over the region of the sample space that satisfies the conditions of the event we are looking for. Our only problem is now to find that region of the correct sample space.

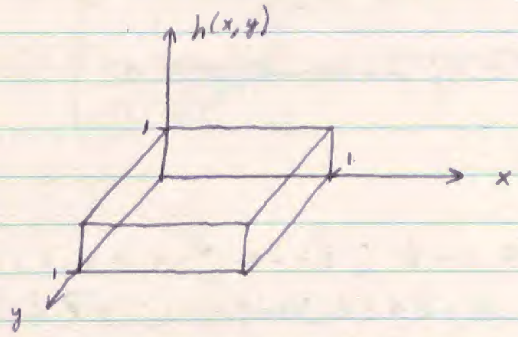
Example:

Cut a stick at 2 places, each cut independent & equally likely at any pt. along the stick:









Now we want to find the probability that we can form a triangle from the 3 pieces of the stick. The lengths of the pieces are:

$y < x$	$y > x$
y	x
$x-y$	$y-x$
$1-x$	$1-y$

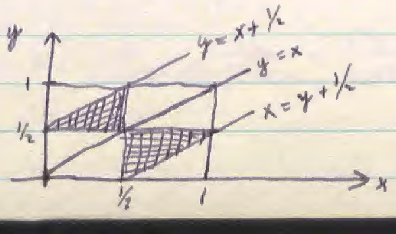
~~By symmetry~~, By symmetry, we see that the two cases $y > x$ & $y < x$ are identical with x & y interchanged, so we need figure the probability for only one case & double it to get the total probability.

The condition that we can form a triangle is that the sum of any two lengths is greater than the third.

$[y > x]$ $\quad (y-x)+(1-y) > x \rightarrow 1-x > x \rightarrow x < \frac{1}{2}$

$\quad x + (y-x) > 1-y \rightarrow y > 1-y \rightarrow y > \frac{1}{2}$

$\quad x + (1-y) > y - x \rightarrow 2x+1 > 2y \quad y < x + \frac{1}{2}$



Similarly for $y < x$

Now, the probability that we can form a triangle is

$$P(\Delta) = \iint_S h(x,y)\,dx\,dy = \iint_S dx\,dy = \text{area}$$

$$\therefore\ P(\Delta) = 2\left(\tfrac{1}{8}\right) = \tfrac{1}{4}$$

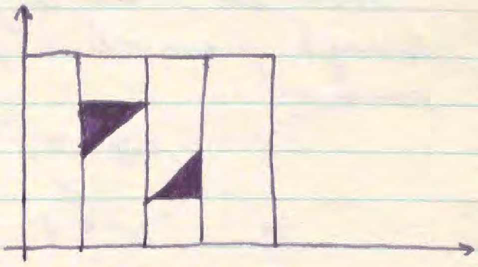
$\boxed{\text{Example \# 2}}$: Find $P(\text{shortest segment} < 1/4)$

We again have symmetry about $y = x$; assume $y > x$

Assume the shortest piece is:

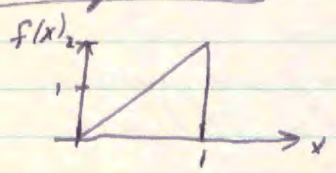
(1) $x \Rightarrow x > \frac{1}{4}$; $y - x > x$; $1 - y > x \Rightarrow x > \frac{1}{4}$; $y > 2x$; $x + y < 1$
(2) $y - x \Rightarrow y - x > \frac{1}{4}$; $x > y - x$; $1 - y > y - x \Rightarrow y > x + \frac{1}{4}$; $2x > y$; $y < \frac{1}{2}(x+1)$
(3) $1 - y \Rightarrow 1 - y > \frac{1}{4}$; $x > 1 - y$; $y - x > 1 - y \Rightarrow y < \frac{3}{4}$; $x + y > 1$; $y > \frac{1}{2}(1+x)$

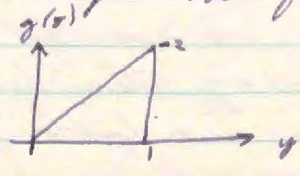
All this reduces to $\quad x > \frac{1}{4}$, $y < 3/4$ & $y > x + \frac{1}{4}$



$$P(<1/4) = 2\left(\tfrac{1}{4} \times \tfrac{1}{4} \times \tfrac{1}{2}\right) = \tfrac{1}{16}$$

$\boxed{\text{Example \#3}}$: Now change $h(x,y)$ & find $P(\Delta)$ again





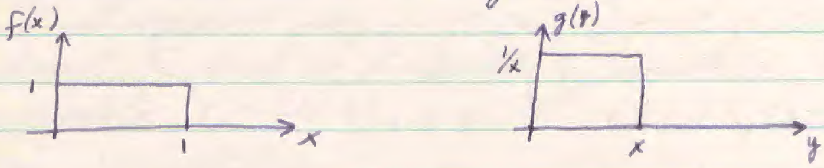
$\rightarrow\ h(x,y) = 4xy \quad 0 < x, y < 1$

The area satisfying the conditions for a triangle is the same, only the density function has changed. Now, we have:

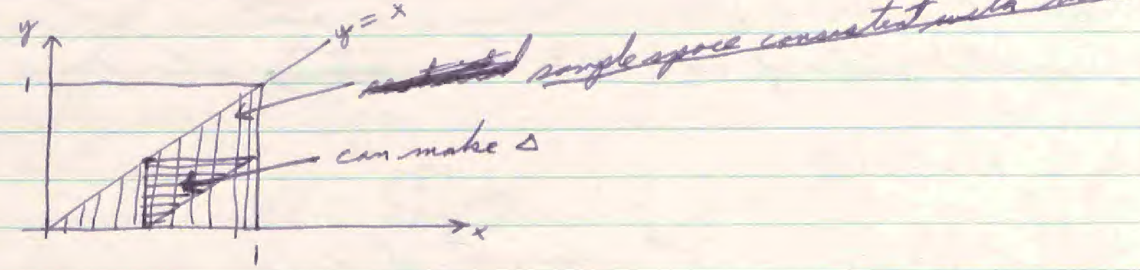
$$P(\Delta) = \iint_S 4xy\,dx\,dy = 2\int_0^{1/2} dx \int_{1/2}^{1/2+x} dy\, 4xy = 2\int_0^{1/2} 2x(x^2 + x)\,dx$$

$$= 2\left[\frac{x^4}{2} + \frac{2x^3}{3}\right]_0^{1/2} = 2\left[\frac{1}{32} + \frac{1}{12}\right] = \frac{11}{48}.$$

<u>Example # 4</u> : Same problem, but cut y on the piece already cut at x.



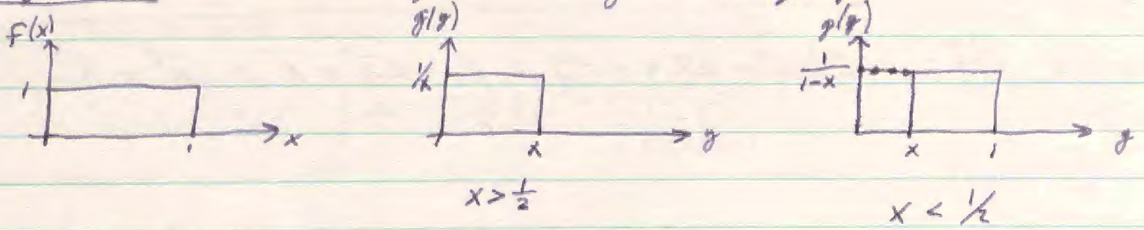
$$h(x,y) = \frac{1}{x} \quad \left\{ \begin{matrix} 0 < x < 1 \\ 0 < y < x \end{matrix} \right\}$$



$$P = \int_0^{1/2} dy \int_{1/2}^{y+1/2} dx \left(\frac{1}{x}\right) = \int_0^{1/2} dy \ln(2y+1) = \frac{1}{2} \int_1^2 du \ln u$$

$$= \frac{1}{2}\Big[u \ln u - u\Big]_1^2 = \frac{1}{2}\Big[2 \ln 2 - 2 + 1 - 0\Big] = \ln 2 - \frac{1}{2} = \underline{.193}$$

$\boxed{\text{Example \#5}}$: Cut at x ; then cut y on longer piece



$$x > \frac{1}{2} \rightarrow h(x,y) = \frac{1}{x} \quad \left\{ \begin{matrix} 0 < x > \frac{1}{2} \\ 0 < y < x \end{matrix} \right\}$$

$$x < \frac{1}{2} \rightarrow h(x,y) = \frac{1}{1-x} \quad \left\{ \begin{matrix} \frac{1}{2} < x < 1 \\ x < y < 1 \end{matrix} \right\}$$

$x > \frac{1}{2}$ is just Example #4 above; $x < \frac{1}{2}$ is the mirror image by symmetry, so

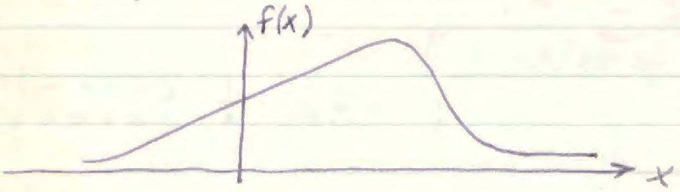
$$P(\Delta) = 2(.193) = 0.386.$$

$\boxed{\text{Example \#6}}$ Find P(chord drawn in circle > side of inscribed triangle)

Chord chosen by : 2 eq. likely pts on circum → P = 1/3
Ruled paper, lines one diameter apart → P = 1/2
Line through drop of rain ⊥ radius → P = 1/4

} P is dependent on what we define <u>exactly</u> as our experiment.

Brief review (HA!) of statistics, now:



$$\int_{-\infty}^{\infty} f(x)\,dx$$

$E(x) \equiv$ expected value of x $= \bar{x} =$ ~~average~~ mean value of x in many trials.

$$E(g[x]) = \int_{-\infty}^{\infty} g(x)\,f(x)\,dx$$

$E(x^k) = \int_{-\infty}^{\infty} x^k f(x)\,dx \equiv$ k^{th} moment of x.

$E(x) = \bar{x} = \int_{-\infty}^{\infty} x\,f(x)\,dx$ = mean value of x = first moment of x.

$E(x^2) = \overline{x^2} = \int_{-\infty}^{\infty} x^2 f(x)\,dx$ = mean square of x = second moment.

$\overline{x^2} - \bar{x}^2 = \sigma^2$; σ = standard deviation of x
σ^2 = variance of x

$$\sigma^2 = E\{(\bar{x}-x)^2\} = E(\bar{x}^2 - 2\bar{x}x + x^2) = \bar{x}^2 - 2\bar{x}^2 + \overline{x^2} = \overline{x^2} - \bar{x}^2$$

Operations on pairs of distribution functions:

Suppose we have $f(x)$ and $g(y)$ given. We now want to find the distribution function $h(z)$ for some new variable $z = \xi(x,y)$; e.g., $z = xy$, $z = x+y$, $z = x^2 y$, etc.

First, we ~~find~~ know the product $f(x)g(y)\,dx\,dy$ (is the prob. that we are in the area ~~[illegible]~~ $dx\,dy$ at x, y.

Now, we make a transformation $z = \xi(x,y)$ on one of our first two variables, say y. Then

$$dz = \left.\frac{\partial \xi}{\partial y}\right|_x dy \quad \text{and} \quad \underline{y = \mathcal{J}(z,x)}. \quad \text{or} \quad \underline{dy = \frac{1}{\left.\frac{\partial\xi}{\partial y}\right|_x} dz}.$$

We can now transform into z, x space with the product

$$f(x)\, g(\mathcal{J}[z,x]) \left| \frac{dz}{\left(\frac{\partial \xi}{\partial y}\right)_x} \right| dx \; \cancel{= h(z)\,dz}$$

as the probability that we are in the area $dz\,dx$ at z, x.

To find the probability that we are at dz, not caring what value x takes on, we integrate out the variable x. So we have:

$$h(z) = \int_{\text{all } x} dx \; f(x)\, g[\mathcal{J}(z,x)] \left| \frac{1}{\left(\frac{\partial \xi}{\partial y}\right)_x} \right|$$

$$\text{or} \quad h(z) = \int_{\text{all } x} dx \; f(x)\, g[y(z,x)] \left| \frac{1}{\left(\frac{\partial z(x,y)}{\partial y}\right)_x} \right|$$

[x & y need not be independent]

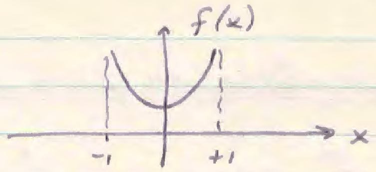
$$\left.\begin{aligned} |x| &= x \\ |-x| &= x \end{aligned}\right\} \text{all } x$$

Note that if ~~$\left(\frac{\partial z}{\partial y}\right)_x$~~ $z(x,y)$ changes in the sign of its slope, we must break the integration into several parts.

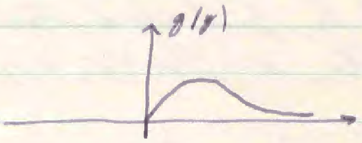
$\boxed{\text{Plot } z \text{ vs } x \text{ to get limits on } x}$

$\boxed{\text{Example}}$

$$f(x) = \frac{1}{\pi\sqrt{1-x^2}} \quad , -1 < x < 1$$

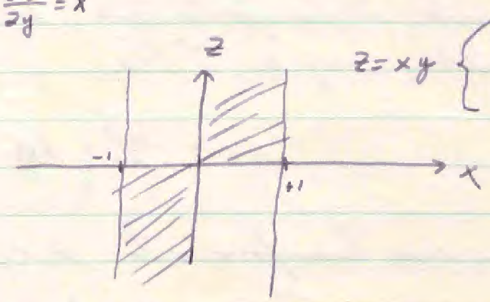


$$g(y) = y\, e^{-y^2/2} \quad , y > 0$$



$\boxed{z = xy}$ $\qquad y = \frac{z}{x} \qquad dz = x\,dy \;;\; \frac{\partial z}{\partial y} = x$

$$h(z) = \int_x f(x)\, g\left(\frac{z}{x}\right) \frac{dx}{|x|}$$



$z = xy \begin{cases} -1 < x < 1 \\ 0 < y < \infty \end{cases}$

Now, $\left.\begin{array}{ll} \underset{(x>0)}{z > 0}, & h(z) = \int_0^1 f(x)\, g\left(\frac{z}{x}\right)\frac{dx}{x} \\ \underset{(x<0)}{z < 0}, & h(z) = +\int_{-1}^0 f(x)\, g\left(\frac{z}{x}\right)\frac{dx}{x} \end{array}\right\}$ $\quad f(x) g\left(\frac{z}{x}\right)\frac{dx}{x} = \frac{1}{\pi\sqrt{1-x^2}}\,\frac{z}{x}\, e^{-\frac{z^2}{2x^2}}\,\frac{dx}{x}$

$$= \frac{z}{\pi\sqrt{1-x^2}}\, e^{-\frac{z^2}{2x^2}}\, \frac{dx}{x^2}$$

Now this is an even function of x, so $+\int_{-1}^0 = \int_0^1$

After much finagling, we get

$$h(z) = \frac{1}{\sqrt{2\pi}}\, e^{-z^2/2} \qquad 0 < z < +\infty$$

$\boxed{\text{NOTE :}}$ $\quad h(x,y) = f(x)\, g(y) \Leftrightarrow x \,\&\, y$ are independent.

If $\quad z = x+y, \qquad y = z-x \qquad dz = dy$

$$\cancel{\quad} \; f(x)\,g(z-x)\,dx\,dz \;=\; P_r\,(\text{in } dx\,dy \text{ at } x,y)$$

$$\boxed{h(z) = \int_{\text{all } x} f(x)\,g(z-x)\,dx \Big]_{z=x+y}} \equiv \underline{\underline{\text{convolution}}}$$

For non-independent ~~and~~ variables

$$h(z) = \int_{\text{all } x} P(x, z-x)\,dx$$

Independence

Def: If $P(AB) = P(A)\,P(B)$, A & B are independent.

This is a mathematical concept & not intuitive.

$\boxed{\text{Example}}$ Flip a coin 3 times ; $P_r(H) = p$

A: at most one tail $\begin{cases} H\,H\,H \\ H\,T\,H \\ H\,H\,T \\ T\,H\,H \end{cases}$

B: all tosses same $\begin{cases} T\,T\,T \\ H\,H\,H \end{cases}$

AB: H H H

$$P(A) = p^3 + 3p^2(1-p)$$

$$P(B) = p^3 + (1-p)^3$$

$$P(AB) = p^3$$

Now $P(AB) \neq P(A)P(B)$ unless $p = 1/2$

Mutual exclusivity $\Rightarrow$ dependence, but not vice versa.

Transforms of probability functions:

Laplace transforms ($f(x) = 0,\ x<0$) or $f(x<0) = 0$

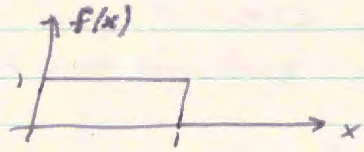
$$\mathcal{L}\{f(x)\} = \int_0^\infty f(x)\, e^{-sx}\, dx = F(s) = E(e^{-sx})$$

$$\left.\frac{\partial F}{\partial s}\right|_{s=0} = -\int_0^\infty x\, f(x)\, e^{-sx}\, dx = -\int_0^\infty x\, f(x)\, dx = -\overline{x}$$

In general, $$\boxed{\left.\frac{\partial^k F}{\partial s^k}\right|_{s=0} = (-)^k\ \overline{x^k}}$$

We can use the short cuts learned in 6.05 to make these transforms easier.

Example:



$F(s) = \frac{1}{s}(1 - e^{-s})$

$$\overline{x} = -\left[\left.\frac{1}{s^2}(1-e^{-s})\right|_{s=0} - \left.\frac{1}{s}(-e^{-s})\right|_{s=0} \right] = -\left.\frac{e^{-s} - 1 + s e^{-s}}{s^2}\right|_{s=0}$$

$$= -\frac{-e^{-s} + e^{-s} - s e^{-s}}{s^2} = +\frac{1}{2} \leftarrow$$

Fourier transforms $\quad -\infty < x < \infty$

$$\mathcal{F}\{f(x)\} = \int_{-\infty}^{\infty} f(x)\, e^{-j\omega x}\, dx = F(j\omega) = E\left(e^{-j\omega x}\right)$$

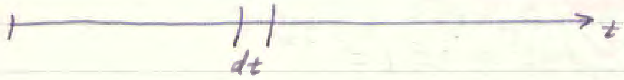
$$\left.\frac{\partial F}{\partial \omega}\right|_{\omega=0} = -j\int_{-\infty}^{\infty} x\, f(x)\, dx = -j\,\overline{x}$$

In general, $\boxed{\overline{x^k} = j^k \dfrac{\partial^k F(j\omega)}{\partial \omega^k}}$

$$\left.\begin{aligned} H(s) &= F(s)\,G(s) \\ H(j\omega) &= F(j\omega)\,G(j\omega) \end{aligned}\right\} \; z = x+y \quad \left[h(z) = f \otimes g\right]$$

Again, we can apply our knowledge of transform + convolution algebra to simplify the work.

Exponential & Poisson distributions:



We assume that we have a number of arrivals distributed so that the probability of an arrival in time dt is $(\lambda\, dt)$.

$P_n(t) = P(n$ arrivals in time $t)$

If $\underline{\underline{n>0}}$, $\quad P_n(t+dt) = P_{n-1}(t)\,\lambda\, dt + P_n(t)\left[1-\lambda\, dt\right]$

or $\quad \dfrac{P_n(t+dt)-P_n(t)}{dt} = \lambda P_{n-1}(t) - \lambda P_n(t)$.

In the limit as $dt \to 0$,

$$\xrightarrow{n>0} \boxed{\frac{dP_n(t)}{dt} = \lambda P_{n-1}(t) - \lambda P_n(t)}$$

If $\underline{\underline{n=0}}$, $\quad P_0(t+dt) = P_0(t)\left[1-\lambda\, dt\right]$

$$\left.\frac{dP_0}{dt} = -\lambda P_0(t)\right\} P_0(t) = Ce^{-\lambda t}$$

Now $P_0(0) = 1$ so $C = 1$

so $\boxed{P_0(t) = e^{-\lambda t}}$

Now, from the differential equation for $n>0$, we can solve for any n:

$\bar{n}(t) = \lambda t$

$$\boxed{P_n(t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}}$$

Poisson distribution
(a function of the discrete variable n)

$\boxed{\text{cf. p. 69}}$

Exponential Distribution

$P_0(t)$ is similar in form to the exponential distribution

$$f(x) = \lambda e^{-\lambda x}. \qquad \bar{x} = \frac{1}{\lambda} \qquad \boxed{F(s) = \frac{\lambda}{s+\lambda}}$$

If we let t now be the time between arrivals, and
$P(t)$ = density of time intervals between arrivals.

$P(t)\,dt$ = probability that the time between arrivals is (t)

$P(t)\,dt = P_0(t)\,\lambda\,dt = P_r[\text{no arrivals up to } t] \times P_r[\text{arrival in } dt]$

$$\boxed{P(t) = \lambda P_0(t) = \lambda e^{-\lambda t}}$$

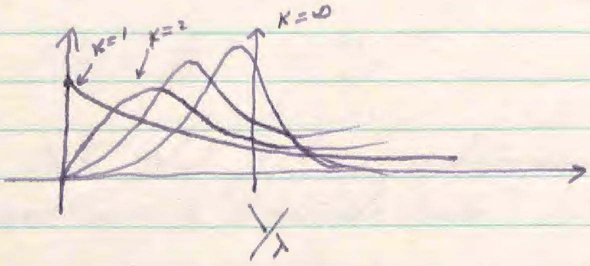
mean $= \sigma = \bar{x}$

If $\lambda \to 2\lambda$, $\quad G(s) \to \left(\frac{2\lambda}{s+2\lambda}\right)^2$

Suppose we take the sum of K samples from an exponential distribution of mean $\frac{1}{K\lambda}$. Now

$$G(s) = \left(\frac{K\lambda}{s+K\lambda}\right)^K \quad \text{for } K \text{ samples}$$

$$\boxed{g(y) = \frac{(K\lambda y)^{K-1}\, e^{-K\lambda y}}{(K-1)!} = \text{Erlang distribution}}$$



This distribution provides a ~~group~~ family of distributions of mean $1/\lambda$ ranging from the most random $(K=1)$ to the most deterministic $(K=\infty \to$ impulse at $1/\lambda)$.
The variance similarly varies over the family.

~~Let y be the probability tha~~

Let y be the time interval between 2 arrivals separated by one arrival

$$G(s) = \left(\frac{\lambda}{s+\lambda}\right)^2$$

$$g(y) = \lambda^2 y e^{-\lambda y}$$

$$\overline{y^0} = 1 \quad - \quad \overline{y} = 2/\lambda \quad ; \quad \overline{y^2} = \frac{6}{\lambda^2}$$

$$\sigma^2 = \frac{2}{\lambda^2}$$

General Notes

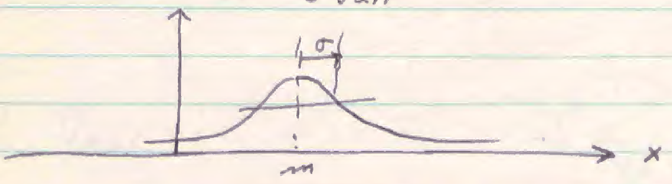
$\overline{X^0} = 1$ is a check that we have an acceptable function

$\sigma^2(x+y) = \sigma^2(x) + \sigma^2(y)$ for x + y independent.

$$\Sigma \overline{X_i} = \overline{\Sigma X_i}$$

Normal Distribution:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{(x-m)^2}{2\sigma^2}} \qquad \begin{cases} m = \text{mean} \\ \sigma = \text{std. deviation} \end{cases}$$



$$-\infty < x < \infty$$

$$F(j\omega) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-j\omega x}\, e^{-\frac{(x-m)^2}{2\sigma^2}}\, dx$$

Let $u = \frac{x-m}{\sigma}$ = standardized variable

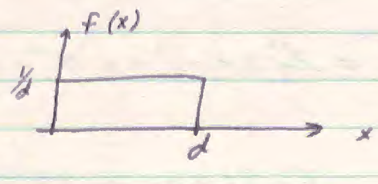
$$F(j\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-j\omega(\sigma u + m)}\, e^{-u^2/2}\, du = \frac{1}{\sqrt{2\pi}} e^{-j\omega m} \int_{-\infty}^{\infty} e^{-u^2/2}\, e^{-j\omega\sigma u}\, du$$

$$= \frac{1}{\sqrt{2\pi}} e^{-j\omega m} \int_{-\infty}^{\infty} e^{-u^2/2} \cos\omega\sigma u\, du = \frac{2}{\sqrt{2\pi}} e^{-j\omega m} \left(\sqrt{\pi}\, \frac{e^{-\frac{\omega^2\sigma^2}{2}}}{2\frac{1}{\sqrt{2}}} \right)$$

$$\boxed{F(j\omega) = e^{-\frac{\omega^2\sigma^2}{2} - j\omega m}}$$

$F(0) = 1$ ✓ ; $\overline{x} = j\frac{\partial F}{\partial \omega}\Big|_0 = m$; $\overline{x^2} = -\frac{\partial^2 F}{\partial \omega^2}\Big|_0 = m^2 + \sigma^2$

Uniform distribution:



$$F(s) = \frac{1}{ds}\left(1 - e^{-ds}\right)$$

Note that $F(0)$ is indeterminant as are the various moments. We could use L'Hopitals rule, but an easier way is to recall that

$$\frac{1}{ds}\left[1 - e^{-ds}\right] = \frac{1}{ds}\left[1 - \left(1 - ds + \frac{(ds)^2}{2!} - \frac{(ds)^3}{3!} + \cdots\right)\right] = 1 - \frac{ds}{2!} + \frac{(ds)^2}{3!} - \frac{(ds)^3}{4!} + \cdots$$

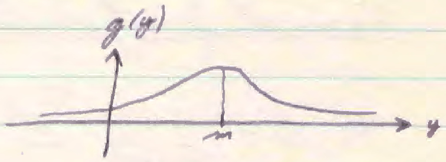
It is now obvious that $F(0) = 1$

$$\left.\begin{aligned} \frac{\partial F}{\partial s}\Big|_{s=0} &= -\frac{d}{2!} \\ \frac{\partial^2 F}{\partial s^2}\Big|_{s=0} &= \frac{d^2}{3} \end{aligned}\right\} \text{etc.}$$

Log-normal distribution:

$$g(y) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{(y-m)^2}{2\sigma^2}}$$

normal



Let $y = \log(x-a)$

$$f(x)\,dx = g(y)\,dy \Rightarrow f(x) = g(y)\frac{dy}{dx}$$

$$\boxed{f(x) = \frac{1}{\sigma\sqrt{2\pi}\,(x-a)} \exp\left\{\frac{-[\ln(x-a)-m]^2}{2\sigma^2}\right\} \quad (x > a)}$$

Now, let $u = x-a$

$$h(u) = \frac{1}{\sigma\sqrt{2\pi}\,u}\, e^{-\frac{[\ln u - m]^2}{2\sigma^2}} \qquad u > 0$$

$$\overline{u^\kappa} = \int_0^\infty u^\kappa h(u)\,du \qquad \& \quad u^\kappa = e^{\kappa \ln u}$$

$$\overline{u^\kappa} = \frac{1}{\sigma\sqrt{2\pi}}\int_0^\infty \frac{du}{u} \exp\left\{\frac{-[-2\kappa\sigma^2\ln u + \ln^2 u - 2m\ln u + m^2]}{2\sigma^2}\right\}$$

now, $(\ln u - m - \kappa\sigma^2)^2 = \ln^2 u + m^2 + \kappa^2\sigma^4 - 2m\ln u - 2\kappa\sigma^2\ln u + 2m\kappa\sigma^2$

so

$$\overline{u^\kappa} = \frac{1}{\sigma\sqrt{2\pi}}\int_0^\infty \frac{du}{u} \exp\left\{-\frac{[(\ln u - m - \kappa\sigma^2)^2 - 2m\kappa\sigma^2 - \kappa^2\sigma^4]}{2\sigma^2}\right\}$$

$$= e^{(\kappa m + \frac{\kappa^2\sigma^2}{2})}\underbrace{\frac{1}{\sigma\sqrt{2\pi}}\int_0^\infty \frac{du}{u} \exp\left\{\frac{-[\ln u - (m+\kappa\sigma^2)]^2}{2\sigma^2}\right\}}$$

This is just the integral over all space of a log-normal distribution of mean $(m+\kappa\sigma^2)$ and as such, integrates to unity.

so $$\boxed{\overline{u^\kappa} = e^{(\kappa m + \frac{\kappa^2\sigma^2}{2})}}$$

$$\overline{u^k} = e^{\left(km + \frac{k^2\sigma^2}{2}\right)} \quad \& \quad \overline{u}^k = e^{\left(km + \frac{k\sigma^2}{2}\right)}$$

so $$\overline{u^k} = \overline{u}^k e^{k(k-1)\frac{\sigma^2}{2}}$$

$$Var(u) = \overline{u^2} - \overline{u}^2 = \overline{u}^2 e^{\sigma^2} - \overline{u}^2 = \overline{u}^2 (e^{\sigma^2} - 1)$$

$$Var(x) = (\overline{x} - a)^2 (e^{\sigma^2} - 1) \quad , \quad u = x - a$$

To obtain moments about different points, we can use the parallel axis theorem of mechanics.

i.e., if $u = x - a$

$$\overline{u} = \overline{x} - a .$$

$$\overline{u} = e^{m + \frac{\sigma^2}{2}}$$

Averaging several samples from distribution :

Consider a normal distribution (m, σ)

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{(x-m)^2}{2\sigma^2}} \quad ; \quad F_x(j\omega) = e^{-\frac{\omega^2\sigma^2}{2} - j\omega m}$$

Now we take n samples from this distribution & sum them.

$$F_{\Sigma x}(j\omega) = \left[F_x(j\omega)\right]^n = e^{-\frac{n\omega^2\sigma^2}{2} - j\omega m n}$$

If $F_y(j\omega) = E(e^{-j\omega y})$

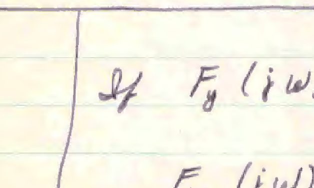
$$F_{y/n}(j\omega) = E(e^{-j\omega \frac{y}{n}}) = E(e^{-j\frac{\omega}{n}y}) = F_y\left(j\frac{\omega}{n}\right)$$

So, for the average of the samples, we have

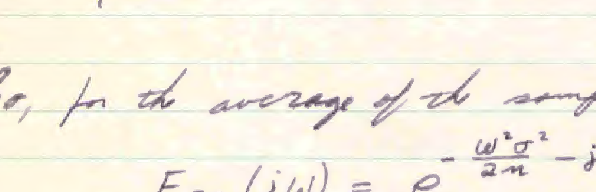
$$F_{\frac{\Sigma x}{n}}(j\omega) = e^{-\frac{\omega^2\sigma^2}{2n} - j\omega m} \leftrightarrow \text{normal}\left(m, \frac{\sigma}{\sqrt{n}}\right)$$

So as n increases, σ decreases. Therefore, as we take more & more samples, our estimate of the mean is better & better.

→ This is not true for all distributions; for example:



A particle emits radiation in all directions. We place a screen ~~in front~~ between it & us & observe the scintillations as the radiation hits the screen:



Now, $g(x) = f(\theta)\,\frac{d\theta}{dx}$

$\tan\theta = x \Rightarrow \frac{dx}{d\theta} = \frac{1}{\cos^2\theta}$ $\qquad -\pi/2 < \theta < \pi/2$

so $g(x) = \frac{1}{\pi}\cos^2\theta = \frac{1}{\pi(1+x^2)}$ $\qquad -\infty < x < \infty$

$$G(j\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-j\omega x}\frac{dx}{1+x^2} = \frac{2}{\pi}\int_0^{\infty}\frac{\cos\omega x}{1+x^2}\,dx = \frac{2}{\pi}\begin{cases}\frac{\pi}{2}e^{-\omega} & ;\ \omega > 0 \\ \frac{\pi}{2}e^{\omega} & ;\ \omega < 0\end{cases}$$

$$\underline{G(j\omega) = e^{-|\omega|}}$$

($G(j\omega)$ exists at $\omega = 0$, but the derivatives do not)

Now, for ~~the~~ an average of n samples, we have

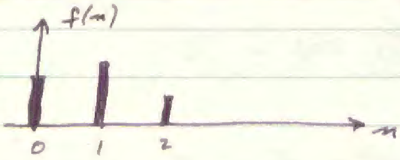
$$H_{\Sigma x}(j\omega) = G^n(j\omega) = e^{-n|\omega|}$$

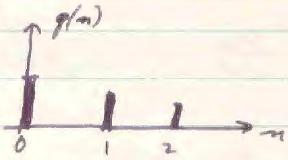
$$H_{\frac{\Sigma x}{n}}(j\omega) = H_{\Sigma x}\left(j\frac{\omega}{n}\right) = e^{-|\omega|} = G(j\omega)$$

Hence, $h(\bar{x}) = g(x)$ and an average of n samples is therefore no better an estimate of the mean ~~where the particle is~~ than is a single observation.

Obviously a long period of observation will let us make a good guess at the mean by relative frequency.

Operational methods for discrete distributions:





$$F(z) \equiv \sum_{n=0}^{\infty} f(n)\, z^{n}$$
$$G(z) \equiv \sum_{n=0}^{\infty} g(n)\, z^{n}$$

Note: $\boxed{F(1) = \sum_{n=0}^{\infty} f(n) = 1}$

If we now take the sum of our samples, $n = n_f + n_g$, we have directly

$$\boxed{h(n) = \sum_{k=0}^{\infty} f(k)\, g(n-k)}$$

$$H(z) = \sum_{n=0}^{\infty} h(n) z^{n} = \sum_{n=0}^{\infty} z^{n} \sum_{k=0}^{\infty} f(k) g(n-k) = \sum_{k=0}^{\infty} f(k) z^{k} \sum_{n=0}^{\infty} g(n-k) z^{n-k}$$

so $\boxed{H(z) = F(z)\, G(z)}$ as for continuous variables.

$$F'(z) = \sum_{n=0}^{\infty} n f(n) z^{n-1} \Rightarrow \boxed{F'(1) = \sum_{n=0}^{\infty} n f(n) = \bar{n}}$$

$$F''(z) = \sum_{n=0}^{\infty} n(n-1) f(n) z^{n-2} \Rightarrow F''(1) = \sum_{n=0}^{\infty} n^2 f(n) - \sum_{n=0}^{\infty} n f(n)$$

so $\boxed{F''(1) = \overline{n^2} - \bar{n}} \Rightarrow \boxed{\overline{n^2} = F''(1) + F'(1)}$

Hence,

$$\boxed{\sigma_n^2 = F''(1) + F'(1) - [F'(1)]^2} \qquad \overline{n^2} = \sigma_n^2 - \bar{n}^2$$

Note that with $z = e^s$ & using an integral transformation of impulses rather than the summation, we have the Laplace transform for an impulse train.

Example: $\left.\begin{array}{l} P_r(1) = p \\ P_r(0) = q \end{array}\right\}$ $p + q = 1$ → Bernoulli process.

$$F(z) = q + pz$$

Now, let $g_K(n) = P_r\{\text{sum of } \underline{K} \text{ samples takes on value } \underline{n}\}$

$$G_K(z) = (F(z))^K = (q + pz)^K$$

$$G_K(z) = q^K + Kpq^{K-1}z + \frac{K(K-1)p^2q^{K-2}}{2!}z^2 + \cdots + p^K z^K = \sum_{n=0}^{K} z^n \binom{K}{n} p^n q^{K-n}$$

$$\therefore \boxed{g_K(n) = \binom{K}{n} p^n q^{K-n} \quad : \text{Binomial distribution}}$$

Now, $G_K(1) = (q+p)^K = 1^K = 1$ ✓

$$G_K'(z) = Kp(q+pz)^{K-1} \longrightarrow \boxed{G_K'(1) = Kp = \overline{n}}$$

$$G_K''(1) = K^2p^2 - Kp^2 \longrightarrow \sigma_n^2 = K^2p^2 - Kp^2 + Kp - K^2p^2 = Kp(1-p) = \boxed{Kpq = \sigma_n^2}$$

Poisson distribution:

$$\boxed{f(n) = \frac{\lambda^n e^{-\lambda}}{n!}} \quad ; \quad F(z) = \sum_{n=0}^{\infty} f(n) z^n = e^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda z)^n}{n!}$$

$$\boxed{F(z) = e^{\lambda z - \lambda} = e^{\lambda(z-1)}} \qquad F(1) = 1 ✓$$

$$F'(z) = \lambda e^{\lambda(z-1)} \rightarrow \boxed{\overline{n} = \lambda} \qquad \sigma_n^2 = \overline{n}$$

$$F''(z) = \lambda^2 e^{\lambda(z-1)} \rightarrow F''(1) = \lambda^2$$

$$\boxed{\sigma_n^2 = \lambda^2 + \lambda - \lambda^2 = \lambda}$$

If we consider two Poisson distributions of means λ_1 & λ_2 respectively, we have for a sum of samples:

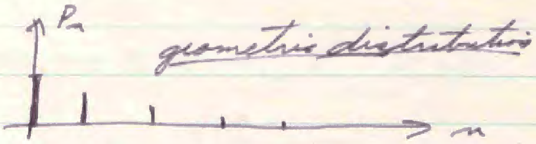
$$H(z) = F(z)G(z) = e^{\lambda_1(z-1)} e^{\lambda_2(z-1)} = e^{(\lambda_1+\lambda_2)(z-1)}$$ which is Poisson.

So the distribution of a sum taken from Poisson distribution is itself Poisson distributed.

Example:

The ~~no~~ number of purchases a customer makes (n) is found to be distributed according to the geometric distribution:

$$P_n = P^n(1-P)$$



This can be modelled by assuming that at any moment, each customer has a probability P of buying another item: so $P_n = P^n(1-P)$.

$$P(z) = \sum_{n=0}^{\infty} P^n(1-P)z^n = (1-P)\sum_{n=0}^{\infty}(Pz)^n = \boxed{\frac{1-P}{1-Pz} = P(z)}$$

$P(1) = 1$ ✓

$$P'(z) = \frac{P(1-P)}{(1-Pz)^2} \rightarrow \boxed{P'(1) = \bar{n} = \frac{P}{1-P}}$$

$$\boxed{\sigma_n^2 = \frac{P}{(1-P)^2}}$$

Now assume the arrival of customers is Poisson distributed with mean λ.

Let $g_m(n) = \Pr\{n \text{ total purchases in time interval } t \text{ in which } m \text{ people arrive}\}$

$h_\lambda(m,t) = \Pr\{m \text{ arrivals occur in time } t\}$

Now we assumed that $h_\lambda(m,t) = \dfrac{(\lambda t)^m e^{-\lambda t}}{m!}$; $g_m(n)$ = dist of sum of m samples from $P^n(1-P)$

$$P(n,t) = \sum_{m=0}^{\infty} g_m(n)\, h_\lambda(m,t)$$

$$P(z,t) = \sum_{n=0}^{\infty} P(n,t)z^n = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} z^n g_m(n)\, h_\lambda(m,t) = \sum_{m=0}^{\infty} G_m(z)\, h_\lambda(m,t)$$

$$= \sum_{m=0}^{\infty}\left(\frac{1-P}{1-Pz}\right)^m \frac{(\lambda t)^m e^{-\lambda t}}{m!} = e^{-\lambda t}\sum_{m=0}^{\infty}\frac{\left[\lambda t\left(\frac{1-P}{1-Pz}\right)\right]^m}{m!}$$

$$\boxed{P(z,t) = e^{-\lambda t}\, e^{\lambda t \frac{1-P}{1-Pz}}} \qquad P(1,t) = 1 \checkmark$$

$$P'(z,t) = e^{-\lambda t} e^{\lambda t \frac{1-p}{1-pz}} \left[\frac{\lambda t (1-p)}{(1-pz)^2} p \right]$$

$$\boxed{P'(1,t) = \lambda t \left(\frac{p}{1-p}\right) = \bar{n}}$$

$$\boxed{\sigma_n^2(t) = \lambda t \frac{p(1+p)}{(1-p)^2}}$$

If everyone made just one purchase, ~~[illegible]~~ the distribution of purchases is just the distribution of arrivals.

$$\left.\begin{array}{l} \sigma_n^2(t) = \lambda t \\ \text{If } p = \tfrac{1}{2}, \quad \sigma_n^2(t) = 3\lambda t \end{array}\right\}$$

$\boxed{\text{Example 2}}$: $\sigma = \Pr\{\text{choose } t \text{ from } 2\sigma\mu e^{-2\sigma\mu t} \quad \text{mean} = \frac{1}{2\sigma\mu}\}$

$1-\sigma = \Pr\{ \quad '' \quad '' \quad '' \quad 2(1-\sigma)\mu e^{-2(1-\sigma)\mu t} \quad \text{mean } \frac{1}{2(1-\sigma)\mu}$

$$f(t) = 2\sigma^2\mu e^{-2\sigma\mu t} + 2(1-\sigma)^2\mu e^{-2(1-\sigma)\mu t} \qquad 0 < t < \infty$$

$$F(s) = \frac{2\sigma^2\mu}{s+2\sigma\mu} + \frac{2(1-\sigma)^2\mu}{s+2(1-\sigma)\mu}$$

$$\left.\begin{array}{l} F(0) = 1 \quad \checkmark \\ \bar{t} = \frac{1}{\mu} \end{array}\right\} \sigma^2(t) = \frac{1-2\sigma+2\sigma^2}{2\sigma(1-\sigma)\mu^2}$$

If $\sigma = \tfrac{1}{2}$, we have a simple experiment with $\sigma^2 = \frac{1}{\mu^2}$

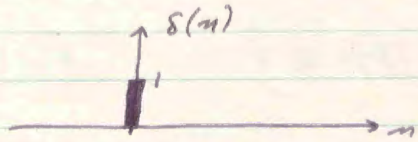
Linear Systems (Discrete systems):

Definition of linearity

If $\left\{ \begin{array}{l} f_1(n) \to g_1(n) \\ f_2(n) \to g_2(n) \end{array} \right\}$ then $a f_1(n) + b f_2(n) = a g_1(n) + b g_2(n)$
for all a & b.

Time invariant: If $f(n) \to g(n)$ then $f(n-k) \to g(n-k)$

Impulse response:



$\delta(n)$ = unit impulse. The response of a system to this input is defined as the impulse response $h(n)$.

For physically realizable systems, $h(n<0) = 0$ so long as we consider n as time. For other applications, where negative n is allowable, we will not have this restriction.

~~To find the outp~~

Response to arbitrary input:

To find the output of a system whose impulse response is $h(n)$ when the input is specified to be $f(n)$, we obviously convolve f & h; i.e.,

$$g(n) = \sum_{k=0}^{\infty} f(k)\, h(n-k) \qquad \equiv \underline{\text{convolution of } f \,\&\, h \equiv f(n) \otimes h(n)}$$

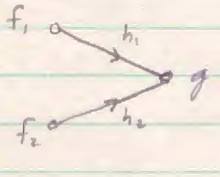
We can, therefore, find the response of the system to any arbitrary input by simply knowing the impulse response. Hence, $h(n)$ completely specifies the system and we need specify only $h(n)$ to " " " ". A symbolic notation is:

$$\circ \xrightarrow{h(n)} \circ$$

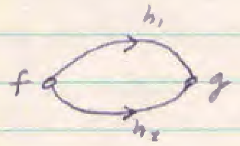
Properties of flow graphs with $h(n)$ specified:

$\circ \xrightarrow{\delta(n)} \circ$ (identity branch) output is just reproduction of input.

$\circ \xrightarrow{\delta(n-k)} \circ$ (delay) output is input delayed in time by k units.



$g(n) = g_1(n) + g_2(n)$ so inputs to a node add.



$g(n) = f \otimes h_1 + f \otimes h_2 = f \otimes (h_1 + h_2)$

so branches in parallel add impulse responses:

$f \circ \xrightarrow{h_1 + h_2} \circ g$

$f \circ \xrightarrow{h_1} \underset{e}{\circ} \xrightarrow{h_2} \circ g$

$$g(n) = \sum_{l=0}^{\infty} e(l)\, h_2(n-l)$$

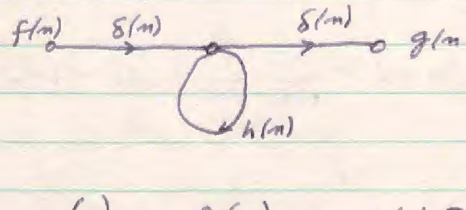
$$= \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} f(k)\, h_1(l-k)\, h_2(n-l)$$

$$= f(n) \otimes \big[h_1(n) \otimes h_2(n) \big]$$

So for branches in series, we convolve impulse responses.

$f \circ \xrightarrow{h_1 \otimes h_2} \circ g$

Example of flow graph solution



$$g(n) = f(n) + g(n) \otimes h(n)$$

$$g(n) \otimes \big[\delta(n) - h(n) \big] = f(n)$$

Here we are stuck as we don't know the reciprocal process for convolution. So, we go to transforms $\longrightarrow$

$\boxed{\text{Z transforms}}$:

$$F(z) = \sum_{n=0}^{\infty} f(n)\, z^{n}$$

If $f(n) = \delta(n)$, then $F(z) = 1$

If $f(n) = a^n,\ n \geq 0$ then $F(z) = \Sigma (az)^n = \dfrac{1}{1-az}$

[If $a = 1$, we have a unit step, $F(z) = \dfrac{1}{1-z}$]

If $f(n) = n a^n,\ n \geq 0$ $\quad F(z) = \sum_{n=0}^{\infty} n\, a^n z^n$

Now if $G(z) = \sum_{n=0}^{\infty} a^n z^n$ then $g(n) = a^n$ & $G(z) = \dfrac{1}{1-az}$

$$G'(z) = \sum_{n=0}^{\infty} n\, a^n z^{n-1} = \frac{a}{(1-az)^2}$$

So $\quad F(z) = \sum_{n=0}^{\infty} n a^n z^n = z\, G'(z) = \dfrac{az}{(1-az)^2}$

[Now if $a = 1$, $f(n) = n$ = unit ramp, $F(z) = \dfrac{z}{(1-z)^2}$]

In general, $f(n-k) \leftrightarrow z^k F(z)$

$\boxed{\text{Convolution of functions}}$

If $g(n) = f(n) \otimes h(n)$ $\qquad$ let $n - k = l$

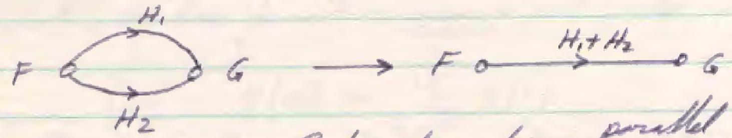
$$G(z) = \sum_{n=0}^{\infty} g(n)\, z^n = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} f(k)\, h(n-k) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} f(k)\, h(l)\, z^k z^l$$

$$= \sum_{k=0}^{\infty} f(k) z^k \sum_{l=0}^{\infty} h(l) z^l = F(z)\, H(z)$$

$$\boxed{G(z) = F(z)\, H(z)}$$

Specification of flow graphs with system functions or transforms:

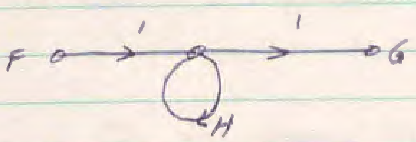
If we know $h(n)$, we can find $H(z)$; so specifying $H(z)$ for a branch specifies that branch completely.



So for branches in ~~series~~ parallel we add transforms just like we added impulse responses.

So for branches in series, we multiply transforms. This is far easier than convolving impulse responses.

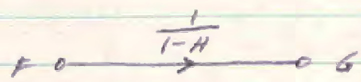
$\boxed{\text{Feedback loop, or self-loop}}$:

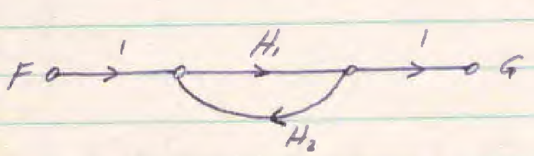


$G = F + GH \rightarrow G(1-GH) = F$

but we can now divide, so

$$\boxed{G = \frac{F}{1-H}}$$

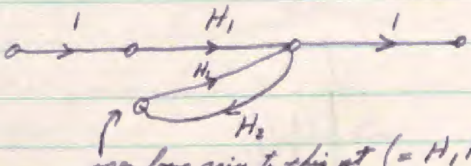




$F\,[1+GH_2]\,H_1 = G$

or $\boxed{G = \frac{H_1}{1-H_1H_2} F}$

We can look at this more physically in the following way



open loop gain to this pt $(= H_1H_2)$

Now $G = H_1\left(\frac{1}{1-H_1H_2}\right) F = \frac{H_1}{1-H_1H_2} F$

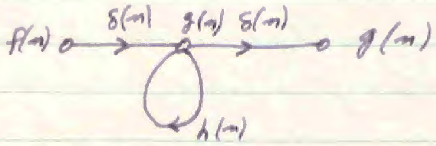
Example

Sales ~~per~~ each month $= f(n)$
Production " " $= g(n)$

$\Big\}$ let $f(n) = \left(\frac{1}{2}\right)^n$

Now to compensate for sales, mgt decides that ~~next~~ production for the month will be sales for the month plus 1/3 last month's production.

This can be attacked as the following system



$h(n) = \frac{1}{3}\delta(n-1)$

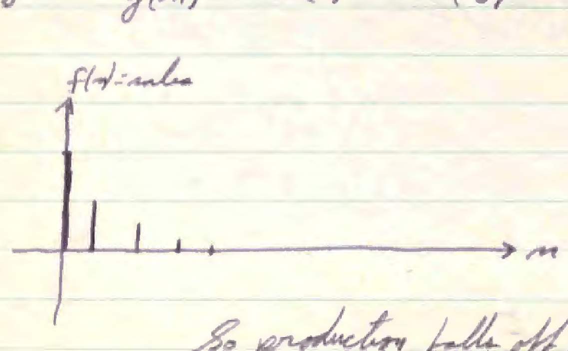
since $g(n) = f(n) + \frac{1}{3}g(n-1)$ or $g(n) \circledast \left[\delta(n) - \frac{1}{3}\delta(n-1)\right] = f(n)$

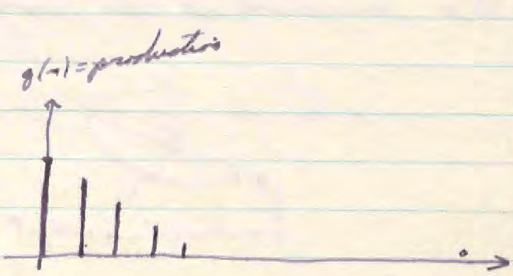
By transforms, $G(z) = \dfrac{F(z)}{1 - H(z)}$

$$\left.\begin{aligned} F(z) &= \frac{1}{1-\frac{z}{2}} \\ H(z) &= \tfrac{1}{3}z = \frac{z}{3} \end{aligned}\right\} \quad G(z) = \frac{1}{\left(1-\frac{z}{2}\right)\left(1-\frac{z}{3}\right)}$$

$$\boxed{G(z) = \frac{3}{1-z/2} - \frac{2}{1-z/3}}$$

So $g(n) = 3\left(\frac{1}{2}\right)^n - 2\left(\frac{1}{3}\right)^n$

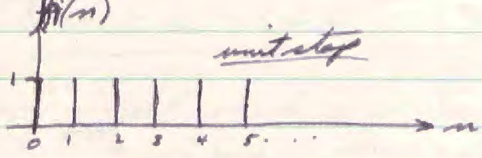




So production falls off as sales falls off, as mgt. wants.

Examples

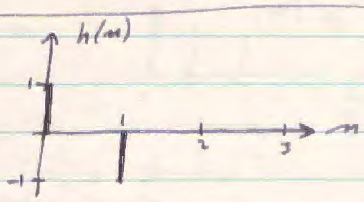
$$\boxed{H(z) = \frac{1}{1-z}} \longleftrightarrow$$



This system function **sums** all past inputs of the input $f(n)$

i.e. $g(n) = \sum_{k=0}^{n} f(k)$

$$\boxed{H(z) = 1 - z} \longleftrightarrow$$

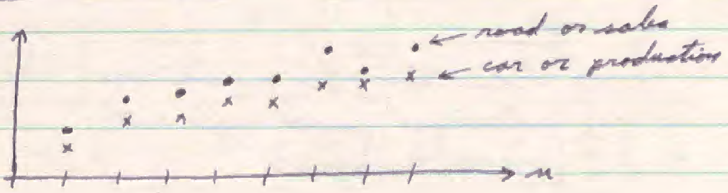


$g(n) = f(n) - f(n-1)$;

i.e., this system function takes the **difference** between successive inputs.

Note that $H(z) = \frac{1}{1+z}$ oscillates as $f(n) = (-1)^n$

Position or ~~[illegible]~~ production control system



No foresight; want car (or production) to follow roadway.

$r^*(n)$ = roadway position at time n
$a^*(n)$ = car " " " "
$e(n)$ = error at time n = $r^*(n) - a^*(n)$
$a(n)$ = change of car position between n + $(n-1)$
$r(n)$ = " " road " " " "

$a^*(n) = a^*(n-1) + a(n)$

$r^*(n) = r^*(n-1) + r(n)$

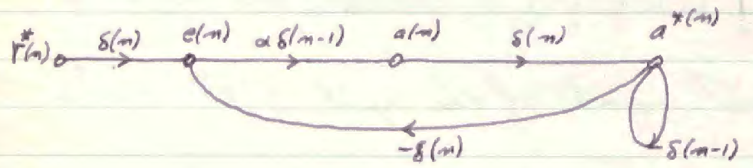
$s^*(n)$ = cumulative sales up to time n
$p^*(n)$ = " production " " "
$i(n)$ = inventory shortage = $s^*(n) - p^*(n)$
$p(n)$ = change in production between n + $(n-1)$
$s(n)$ = " " sales " " "

$p^*(n) = p^*(n-1) + p(n)$

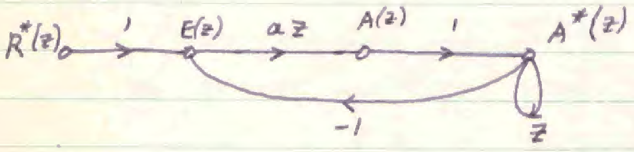
$s^*(n) = s^*(n-1) + s(n)$

Let us say that $\underline{a(n) = \alpha e(n-1)}$ or $P(n) = \alpha i(n-1)$

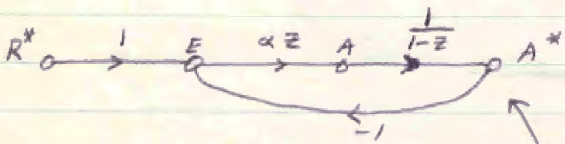
$\underline{a^*(n) = a^*(n-1) + a(n)}$



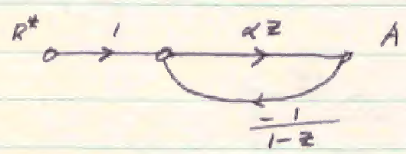
$\Leftarrow$ by inspection



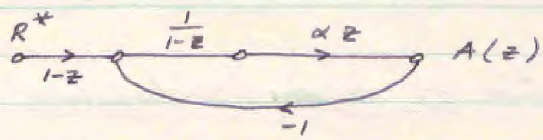
The self-loop on $A^*(z)$ can be eliminated by placing a branch with $H(z) = \frac{1}{1-z}$ in all branches coming $\underline{\text{into}}$ $A^*(z)$



If we choose to solve for $A(z)$ we can redraw the above as:



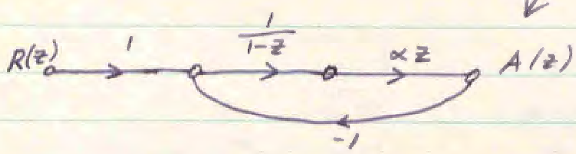
or



But $R^*(z) \xrightarrow{1-z} R(z)$

These two flow graphs are identical, with different inputs & outputs

so



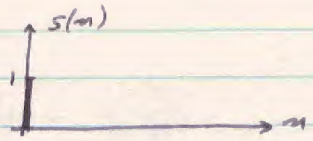
$$\frac{A(z)}{R(z)} = \frac{A^*(z)}{R^*(z)} = \frac{\frac{\alpha z}{1-z}}{1+\frac{\alpha z}{1-z}} = \frac{\alpha z}{1-(1-\alpha)z} = \frac{P(z)}{S(z)} = \frac{P^*(z)}{S^*(z)}$$

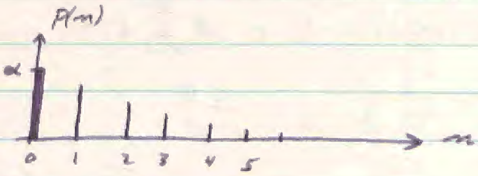
$$\frac{E(z)}{R(z)} = \frac{1}{1-(1-\alpha)z} = \frac{I(z)}{S(z)}$$

Now, let's tickle the system:

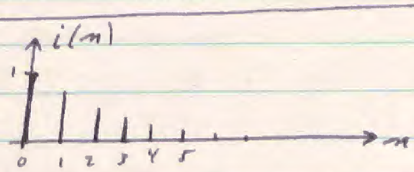
Let $\boxed{S(z) = 1}$ so we have an input which is an impulse of sales:

$$P(z) = \frac{\alpha z}{1-(1-\alpha)z} \longrightarrow P(n) = \boxed{\alpha(1-\alpha)^n} \qquad (1-\alpha) < 1 \text{ for stability}$$





$$\boxed{I(z) = \frac{1}{1-(1-\alpha)z} \longrightarrow \boxed{i(n) = (1-\alpha)^n}}$$



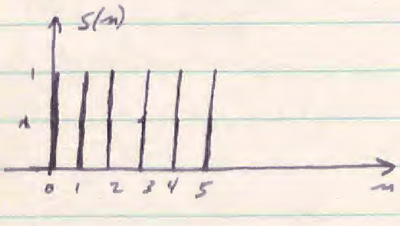
Now apply a step in sales; $\boxed{S(z) = \frac{1}{1-z}}$

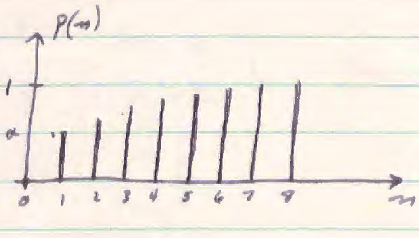
$$P(z) = \left(\frac{1}{1-z}\right)\left(\frac{\alpha z}{1-(1-\alpha)z}\right) = \frac{1}{1-z} + \frac{-1}{1-(1-\alpha)z}$$

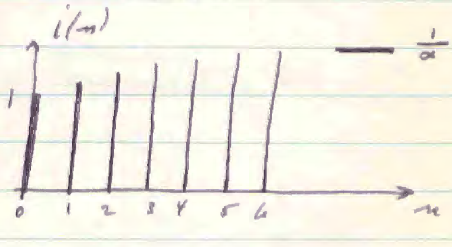
$$\boxed{P(n) = 1 - (1-\alpha)^n}$$

$$I(z) = \left(\frac{1}{1-z}\right)\left(\frac{1}{1-(1-\alpha)z}\right) = \frac{1/\alpha}{1-z} + \frac{-\left(\frac{1-\alpha}{\alpha}\right)}{1-(1-\alpha)z}$$

$$\boxed{i(n) = \frac{1}{\alpha} - \frac{1-\alpha}{\alpha}(1-\alpha)^n = \frac{1}{\alpha}\left[1-(1-\alpha)^{n+1}\right]} = \frac{1}{\alpha}P(n+1)$$





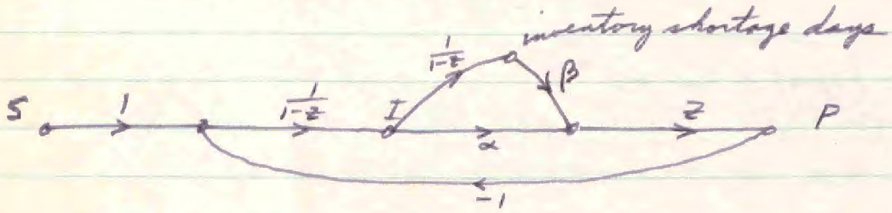


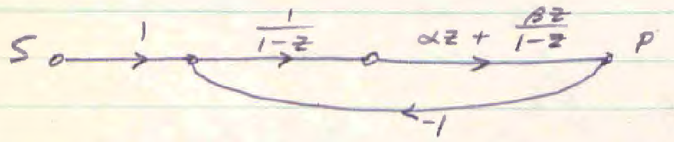
Variation of example

Let us now change the way we adjust production to inventory. Up to now we have taken a fixed fraction of $i(n)$ as our production. But as the last example (a step of sales) shows, we are not accounting for the inventory shortage we carry along day-to-day. To step up production enough to eliminate this effect, we set

$$P(n) = \alpha\, i(n-1) + \beta \sum_{k=0}^{n-1} i(k).$$

$$P(z) = \alpha z + \frac{\beta z}{1-z}$$





$$\frac{P(z)}{S(z)} = \frac{\frac{1}{1-z}\left[\alpha z + \frac{\beta z}{1-z}\right]}{1 + \frac{1}{1-z}\left[\alpha z + \frac{\beta z}{1-z}\right]} = \frac{z\,(\alpha+\beta-\alpha z)}{1+(\alpha+\beta-2)z+(1-\alpha)z^2}$$

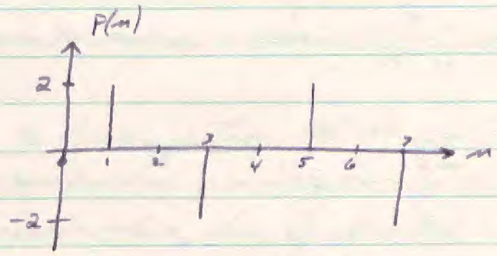
$$\frac{I(z)}{S(z)} = \frac{\frac{1}{1-z}}{1+\frac{1}{1-z}\left[\alpha z+\frac{\beta z}{1-z}\right]} = \frac{1-z}{1+(\alpha+\beta-2)z+(1-\alpha)z^2}$$

Example of ~~example~~ variation of example: $\begin{cases} \alpha = 0 \\ \beta = 2 \end{cases}$

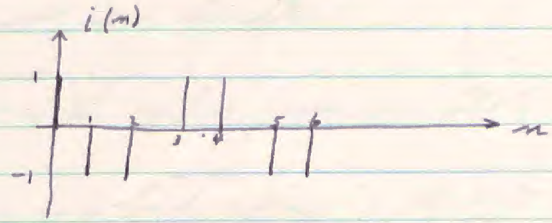
$$\left.\begin{aligned} \frac{P}{S} &= \frac{2z}{1+z^2} \\[2em] \frac{I}{S} &= \frac{1-z}{1+z^2} \end{aligned}\right\}$$

Let $S(z) = 1$ (impulse of sales)

$P(z) = \frac{2z}{1+z^2}$

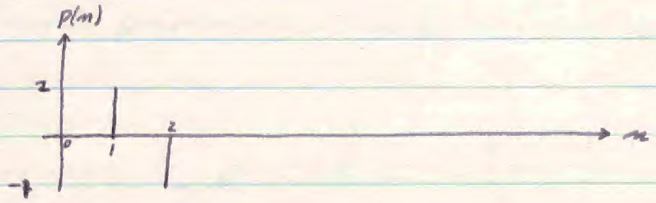


$I(z) = \frac{1-z}{1+z^2}$



$\alpha = \beta = 1$, $S = 1$

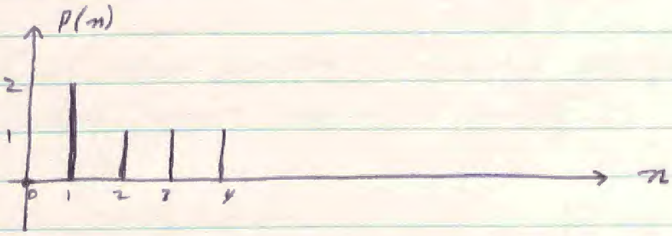
$P(z) = 2z - z^2$



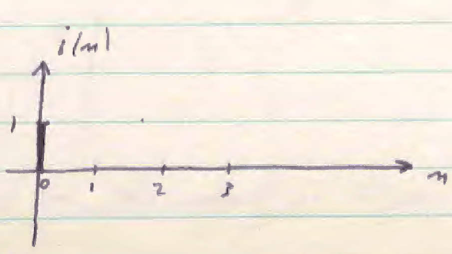
$I(z) = 1 - z$

Now let $\alpha = \beta = 1$ & $S = \frac{1}{1-z} \leftrightarrow$ step of sales

$P(z) = \frac{2z - z^2}{1-z}$



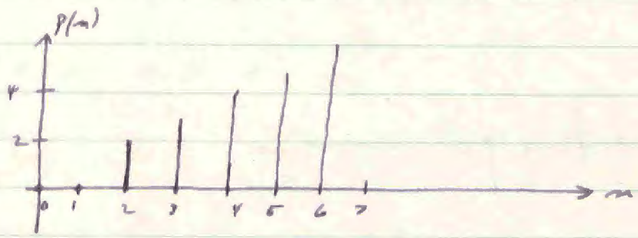
$I(z) = 1$



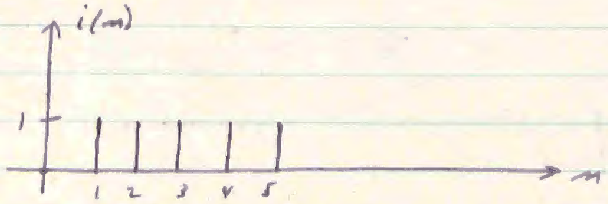
Example of second variation of example:

Let $\alpha = \beta = 1$, $S = \frac{z}{(1-z)^2} \leftrightarrow$ ramp of sales

$$P(z) = (2z - z^2)\frac{z}{(1-z)^2} = 2z\frac{z}{(1-z)^2} - z^2\frac{z}{(1-z)^2}$$



$$I(z) = \frac{z}{1-z}$$



Markov Processes:

A Markov process is a mathematical model of a system.

State of a system $\equiv$ a description of the system in terms of several variables; necessarily not a complete description, due to limited numbers of variables we can handle

Transition: change of the system from one state to another

Discrete time systems: regularly spaced intervals between transitions but not necessarily time intervals.

Continuous time: intervals probabilistically determined.

We restrict our attention for now to discrete time systems.

Examples:

Frog jumping between several numbered lilly pads.

$p_{ij} = \Pr\{$ frog on i will jump to j in his next jump $\}$

$$\sum_{j=1}^{N} p_{ij} = 1 \quad ; \quad 0 \le p_{ij} \le 1$$

We can now represent all these probabilities in the probability matrix $P \equiv [p_{ij}]$ whose rows sum to unity.

Taxi driver operating in towns 1 & 2.

$$P = \begin{bmatrix} 3/4 & 1/4 \\ 1/3 & 2/3 \end{bmatrix}$$

p_{11} = Pr will go to town 1 if in 1
p_{12} = " " " " " 2 " " 1
p_{21} = " " " " " " 1 " " 2

Toymaker (1) has successful toy this week or (2) has a flop this week.

$$P = \begin{bmatrix} 1/2 & 1/2 \\ 2/5 & 3/5 \end{bmatrix}$$

State probabilities of a system:

$\pi_i(n) \equiv$ probability that the system will be in state i at time n $(n = 0, 1, 2, 3, \ldots)$ for a given starting at $n = 0$.

$$\sum_{i=1}^{N} \pi_i(n) = 1 \qquad n = 0, 1, 2, \ldots$$

$$\boxed{\pi_j(n+1) = \sum_{i=1}^{N} \pi_i(n) P_{ij} \qquad j = 1, 2, \ldots, N}$$

Now we go to a vector or matrix representation of the state probabilities:

$$\underline{\pi}(n) = [\pi_1(n) \quad \pi_2(n) \cdots \quad \pi_N(n)]$$

So we can write the above recursion formulas in the form:

$$\boxed{\underline{\pi}(n+1) = \underline{\pi}(n) P}$$

$$\underline{\pi}(1) = \underline{\pi}(0) P \quad ; \underline{\pi}(2) = \underline{\pi}(1) P = \underline{\pi}(0) P^2 \quad ; \text{etc, so}$$

$$\boxed{\underline{\pi}(n) = \underline{\pi}(0) P^n}$$ c.f. p. 87

Example: Toymaker with given probability matrix (p. 41)

$$\pi_1(0) = 1 \quad ; \quad \pi_2(0) = 0 \Rightarrow \underline{\pi}(0) = [1 \quad 0]$$ } i.e., start with a success.

$$\underline{\pi}(1) = \underline{\pi}(0) P = [1 \quad 0] \begin{bmatrix} 1/2 & 1/2 \\ 2/5 & 3/5 \end{bmatrix} = [1/2 \quad 1/2]$$

$$\underline{\pi}(2) = \pi(1) P = [1/2 \quad 1/2] \begin{bmatrix} 1/2 & 1/2 \\ 2/5 & 3/5 \end{bmatrix} = [9/20 \quad 11/20]$$

etc.

or we could write directly $P^2 = \begin{bmatrix} \frac{45}{100} & \frac{55}{100} \\ \frac{44}{100} & \frac{66}{100} \end{bmatrix}$

Another way to go at the problem is to draw up a table of $\pi_i(n)$ for $n = 0, 1, 2, \ldots$

$n \rightarrow$	0	1	2	3	4	5
$\pi_1(n)$	1	.5	.45	.445	.4445	
$\pi_2(n)$	0	.5	.55	.555	.5555	

$$\lim_{n\to\infty} \pi_1(n) = 4/9 \qquad \lim_{n\to\infty} \pi_2(n) = 5/9$$

Now suppose we started in the other state $\underline{\pi}(0) = [0 \ 1]$

$n \rightarrow$	0	1	2	3	
$\pi_1(n)$	0	.4	.44	.444	
$\pi_2(n)$	1	.6	.56	.556	

again, the state probabilities converge to the same values as before, independent of starting state.

Ergodic process: one in which the state probabilities approach a limit.

Completely ergodic process: the limits of the state probabilities are independent of initial state.

We now restrict our consideration to completely ergodic process.

$$\boxed{\text{Define } \lim_{n\to\infty} \pi_i(n) \equiv \pi_i \quad \& \quad \sum_{i=1}^{N} \pi_i = 1}$$

$$\boxed{\text{Now } \lim_{n\to\infty} \underline{\pi}(n+1) = \lim_{n\to\infty} \underline{\pi}(n) \equiv \underline{\pi}}$$

so $\boxed{\underline{\pi} = \underline{\pi}\, \underline{P}}$ where $\underline{\pi} \equiv [\pi_1 \ \pi_2 \ \ldots \ \pi_N]$

Example: Toymaker

$$\underline{\pi} = \underline{\pi}\, P \Rightarrow \left\{ \begin{array}{l} \frac{1}{2}\pi_1 + \frac{2}{5}\pi_2 = \pi_1 \\ \frac{2}{5}\pi_1 + \frac{3}{5}\pi_2 = \pi_2 \end{array} \right\} \Rightarrow \underline{\underline{\pi_1 = \tfrac{4}{5}\pi_2}}$$

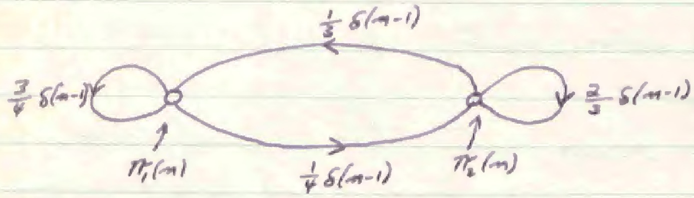
$$\left.\begin{array}{l} \pi_1 = \frac{4}{5}\pi_2 \\ \pi_1 + \pi_2 = 1 \end{array}\right\} \Rightarrow \begin{array}{l} \pi_1 = 4/9 \\ \pi_2 = 5/9 \end{array}$$

The rank of the matrix $\underline{\pi} = \underline{\pi}\, P$ is $N-1$

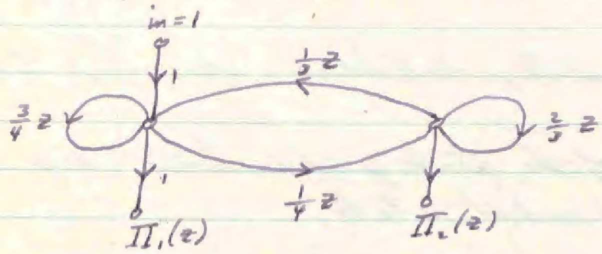
Flow graphs for Markov processes:

Consider the taxi driver with states (1) and (2)

$$\left.\begin{aligned} \pi_1(n) &= \tfrac{3}{4}\pi_1(n-1) + \tfrac{1}{3}\pi_2(n-1) \\ \pi_2(n) &= \tfrac{1}{4}\pi_1(n-1) + \tfrac{2}{3}\pi_2(n-1) \end{aligned}\right\} \text{ from } \underline{\pi}(n) = \pi(n-1)\,P$$



Or, using transforms to the land of Oz,



$$\left.\begin{aligned} \Pi_1(z) &= \tfrac{3}{4}z\,\Pi_1(z) + \tfrac{1}{3}z\,\Pi_2(z) \\ \Pi_2(z) &= \tfrac{1}{4}z\,\Pi_1(z) + \tfrac{2}{3}z\,\Pi_2(z) \end{aligned}\right.$$

Using our old formula for flow graph transmission, ~~we get~~ and starting the system off with an impulse in at state (1) corresponding to $\underline{\pi}(0) = [1\,0]$, we get

$$\Pi_1(z) = \frac{1 - \frac{2}{3}z}{1 - \frac{3}{4}z - \frac{2}{3}z - \frac{1}{12}z^2 + \frac{1}{2}z^2} = \frac{1 - \frac{2}{3}z}{(1-z)(1-\frac{5}{12}z)}$$

For a realizable system, we must be able to factor out $(1-z)$ in the denominator corresponding to convolution with a unit step.

$$\Pi_1(z) = \frac{4/7}{1-z} + \frac{3/7}{1-\frac{5}{12}z}$$

$$\pi_1(n) = \frac{4}{7} + \frac{3}{7}\left(\frac{5}{12}\right)^n$$

Similarly,

$$\Pi_2(z) = \frac{\frac{1}{4}z}{(1-z)(1-\frac{5}{12}z)} = \frac{3/7}{1-z} - \frac{3/7}{1-\frac{5}{12}z}$$

$$\pi_2(n) = \frac{3}{7} - \frac{3}{7}\left(\frac{5}{12}\right)^n$$

$\pi_i(n)$ can always be broken into two parts $\begin{cases} \text{steady state} \\ \text{transient (function of } n\text{)} \end{cases}$

We will never have instability problems, but the transient could oscillate rather than converge as the above does.

Matrix Transforms:

Recalling some characteristics of function transforms

$$f(n) \leftrightarrow F(z)$$
$$f(n-1) \leftrightarrow z F(z)$$
$$f(n+1) \leftrightarrow z^{-1}[F(z) - f(0)]$$

we see that $\quad \underline{\Pi}(n+1) = \underline{\pi}(n)P$, gives

$$z^{-1}[\underline{\Pi}(z) - \underline{\pi}(0)] = \underline{\Pi}(z)P$$

$$\underline{\Pi}(z)[I - zP] = \underline{\pi}(0)$$

~~$\underline{\Pi}(z) = \underline{\pi}(0)$~~ $\qquad \boxed{\underline{\Pi}(z) = \underline{\pi}(0)[I - zP]^{-1}}$

Now, define $H(n)$ to be the inverse transform of the matrix $[I - zP]^{-1}$

$$\boxed{\underline{\pi}(n) = \underline{\pi}(0)\,H(n)}$$

We now have a closed form for $P^n = H(n)$. The elements of $H(n)$, [illegible] are such that $h_{ij}(n)$ is the probability that the system will occupy state j at time n, given that it occupied state i at $n=0$.

<u>$H(n)$ is a complete description of the system.</u>

Taxi example:

$$P = \begin{bmatrix} 3/4 & 1/4 \\ 1/3 & 2/3 \end{bmatrix} \text{ so } [I - zP] = \begin{bmatrix} 1-\frac{3}{4}z & -\frac{1}{4}z \\ -\frac{1}{3}z & 1-\frac{2}{3}z \end{bmatrix}$$

$$\boxed{\det[I - zP] = (1-\tfrac{5}{12}z)(1-z) \equiv \text{denominator of flow graph of system}}$$

$$[I - zP]^{-1} = \frac{1}{(1-z)(1-\frac{5}{12}z)} \begin{bmatrix} 1-\frac{2}{3}z & \frac{1}{4}z \\ \frac{1}{3}z & 1-\frac{3}{4}z \end{bmatrix}$$

Doing a partial fraction expansion element by element gives

$$[I - zP]^{-1} = \begin{bmatrix} \frac{4/7}{(1-z)} + \frac{3/7}{(1-\frac{5}{12}z)} & \frac{3/7}{1-z} + \frac{-3/7}{1-\frac{5}{12}z} \\ \frac{4/7}{1-z} + \frac{-4/7}{1-\frac{5}{12}z} & \frac{3/7}{1-z} + \frac{4/7}{1-5/12\,z} \end{bmatrix}$$

$$[I - zP]^{-1} = \frac{1}{1-z}\begin{bmatrix} 4/7 & 3/7 \\ 4/7 & 3/7 \end{bmatrix} + \frac{1}{1-\frac{5}{12}z}\begin{bmatrix} 3/7 & -3/7 \\ -4/7 & 4/7 \end{bmatrix}$$

Transforming this, we can write

$$H(n) = \begin{bmatrix} 4/7 & 3/7 \\ 4/7 & 3/7 \end{bmatrix} + \left(\frac{5}{12}\right)^n \begin{bmatrix} 3/7 & -3/7 \\ -4/7 & 4/7 \end{bmatrix} \equiv S + T(n)$$

We can always write $\boxed{H(n) = S + T(n)}$

$S \equiv$ limiting state probability matrix; it is a stochastic matrix, i.e., its rows each sum to unity.

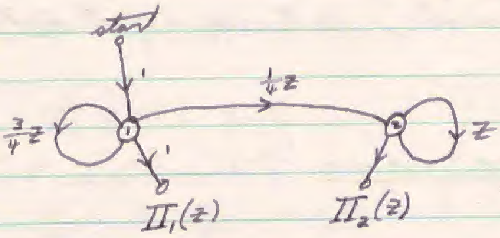
$T(n) \equiv$ transient state probability matrix; it is a differential matrix, i.e., its rows each sum to zero.

$$\Pi(n) = \Pi(0)\, H(n)$$

The matrix S is the matrix of "steady state" state probabilities for each possible starting position of the system. For example, the 2^{nd} row is the limiting state probability matrix given that the system started off in the 2^{nd} state at $n=0$. Here all rows of S are identical, so $\Pi(n)$ is independent of starting position and $\Pi = [\frac{4}{7} \; \frac{3}{7}]$. If we start such that $\Pi(0) = [4/7 \; 3/7]$, $T(n) \equiv 0$ for all n & $\Pi(n) = \Pi = [4/7 \; 3/7]$.

New probability matrix for the taxi driver:

Let $P = \begin{bmatrix} 3/4 & 1/4 \\ 0 & 1 \end{bmatrix}$.



Once the taxi gets to (2), he will stay there.

State (1) is a transient state; i.e., $\lim_{n\to\infty} \pi_1(n) = 0$. The system will eventually leave this state and never return.

State (2) is a trapping state; i.e., $P_{ii} = 1$. Once the system gets into this state, it will never leave.

Solving the above flow graph, we get

$$\Pi_2(z) = \frac{1/4\, z}{(1-z)(1-\frac{3}{4}z)} = \frac{1}{1-z} + \frac{-1}{1-\frac{3}{4}z}$$

$$\Pi_1(z) = \frac{1}{1-\frac{3}{4}z}$$

$$\pi_1(n) = \left(\tfrac{3}{4}\right)^n$$
$$\pi_2(n) = 1 - \left(\tfrac{3}{4}\right)^n$$

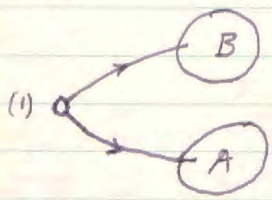
To find the steady state probability directly, we write

$\underline{\pi} = \underline{\pi} P$ where $\pi_i = \lim_{n\to\infty} \pi_i(n)$ & $\sum_i \pi_i = 1$

$$P = \begin{bmatrix} 3/4 & 1/4 \\ 0 & 1 \end{bmatrix} \rightarrow \left.\begin{array}{l} \pi_1 = \frac{3}{4}\pi_1 \rightarrow \pi_1 = 0 \\ \pi_2 = \frac{1}{4}\pi_1 + \pi_2 \\ \pi_1 + \pi_2 = 1 \end{array}\right\} \begin{array}{l} \pi_1 = 0 \\ \pi_2 = 1 \end{array}$$

State (2) in the preceeding example is a simple case of a <u>recurring chain</u>.
A <u>recurring chain</u> is a combination (or group) of states with no outgoing
→ probability. Once the system falls into the chain,
→ it can bounce around inside, but never get out.

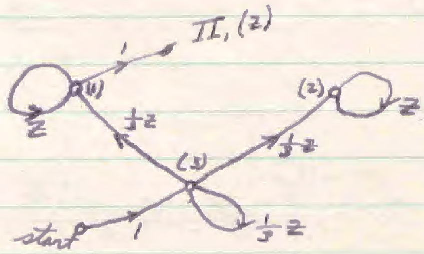
<u>Systems with more than one recurring chain</u>:



(1) is a transient state and A + B are trapping chains (or recurrent chains).

<u>Example</u>:

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$



Since ~~loop~~ chain (2) does not have a branch coming back into the (1) (3) system, we can ignore it for purposes of writing the transmission ~~fcn~~ from (3) to (1).

$$\Pi_1(z) = \frac{\frac{1}{3}z}{(1-z)(1-\frac{1}{3}z)} = \frac{\frac{1}{2}}{1-z} - \frac{\frac{1}{2}}{1-\frac{1}{3}z}$$

But note that $(1-z)(1-\frac{1}{3}z)$ is <u>not</u> the denominator for our flow graph. This must contain <u>all</u> loops, regardless of the particular path we are considering. Here it is $(1-z)(1-z)(1-\frac{1}{3}z)$.

If the system starts off in (3),

$$\Pi_1(n) = \Pi_2(n) = \tfrac{1}{2} - \tfrac{1}{2}\left(\tfrac{1}{3}\right)^n$$
$$\Pi_3(n) = \left(\tfrac{1}{3}\right)^n$$

If the system starts off in (1),

$$\Pi_1(n) = 1$$
$$\Pi_2(n) = \Pi_3(n) = 0$$

This shows that the limiting state probabilities are not independent of where the system started. If we try to solve for this, we get

$$\underline{\Pi} = \underline{\Pi} P, \quad \sum \pi_i = 1$$

$$\left.\begin{aligned} \pi_1 &= \pi_1 + \tfrac{1}{3}\pi_3 \\ \pi_2 &= \pi_2 + \tfrac{1}{3}\pi_3 \\ \pi_3 &= \tfrac{1}{3}\pi_3 \rightarrow \pi_3 = 0 \end{aligned}\right\} \quad \left.\begin{aligned} \pi_1 &= \pi_1 \\ \pi_2 &= \pi_2 \\ \pi_3 &= 0 \end{aligned}\right\} \quad \boxed{\begin{aligned} \pi_1 + \pi_2 &= 1 \\ \pi_3 &= 0 \end{aligned}}$$

When ~~we~~ the limiting state probabilities are no longer independent, we find that we lack information to solve for them by this method. That is, the rank of $\underline{\Pi} = \underline{\Pi} P$ is less than $N-1$.

<u>Now, we try to compute $H(n)$</u>:

$$[I - zP] = \begin{bmatrix} 1-z & 0 & 0 \\ 0 & 1-z & 0 \\ -\frac{1}{3}z & -\frac{1}{3}z & 1-\frac{1}{3}z \end{bmatrix}$$

$$[I - zP]^{-1} = \frac{1}{(1-z)^2(1-\frac{1}{3}z)} \begin{bmatrix} (1-z)(1-\frac{1}{3}z) & 0 & 0 \\ 0 & (1-z)(1-\frac{1}{3}z) & 0 \\ \frac{1}{3}z(1-z) & (1-z)(\frac{1}{3}z) & (1-z)^2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{1-z} & 0 & 0 \\ 0 & \frac{1}{1-z} & 0 \\ \frac{\frac{1}{3}z}{(1-z)(1-\frac{1}{3}z)} & \frac{\frac{1}{3}z}{(1-z)(1-\frac{1}{3}z)} & \frac{1}{1-\frac{1}{3}z} \end{bmatrix}$$

$$= \frac{1}{1-z} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} + \frac{1}{1-\frac{1}{3}z} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix}$$

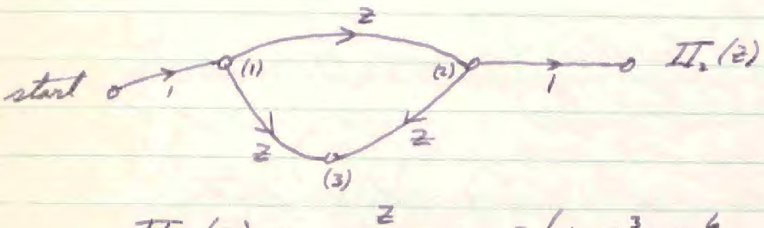
so $$H(n) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} + \left(\frac{1}{3}\right)^n \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix} = S + T(n)$$

Row i of S is $\underline{\Pi}$ given that the system started in state i. This is the same interpretation as before except that all rows are not necessarily the same.

In general, we will have as many transient terms as we have recurrent chains. Here, the "time constants" are the same for both chains (1/3) ~~by~~ symmetry & $T(n)$ is reduced to one term.

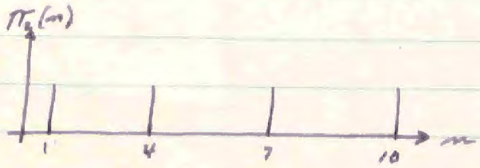
Periodic processes:

A periodic process is one in which we know at any time where the system will be, given a starting position - usually it will close on itself:



$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\Pi_2(z) = \frac{z}{1-z^3} = z\left(1+z^3+z^6+\cdots\right) = z+z^4+z^7+\cdots$$



period = 3

The limiting state probabilities simply do not exist in this case. If we try to find them, using our matrix equation $\underline{\pi} = \underline{\pi}P$, we get

$$\left.\begin{aligned} \pi_1 &= \pi_3 \\ \pi_2 &= \pi_1 \\ \pi_3 &= \pi_2 \\ \pi_1 + \pi_2 + \pi_3 &= 1 \end{aligned}\right\} \quad \underline{\pi_1 = \pi_2 = \pi_3 = \tfrac{1}{3}}$$

We can interpret the π_i's we get in this way by saying that they are the probabilities of finding the system in a given state if the time is picked at random.

Finding $[I - zP]^{-1}$ from the flow graph:

$$\left([I - zP]^{-1}\right)_{ij} = \frac{(\text{numerator})_{ij}}{\det[I - zP]}$$

We can readily find $\det[I - zP]$ from the P matrix. If we can find the numerator from the flow graph, we have found $[I - zP]^{-1}$. Suppose we write the above equation as

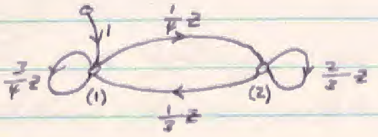
$$(\text{numerator})_{ij} = \det[I - zP]\,[I - zP]^{-1}_{ij}$$

or $$(\text{output}_{ij}) = (\text{input}) \times (\text{transfer function})_{ij}$$

Thus, if we insert an input $\det[I - zP]$ at node i and take the output at node j, we get the matrix element $\left([I - zP]^{-1}\right)_{ij}$.

$\boxed{\text{Example}}$ $\quad P = \begin{bmatrix} 3/4 & 1/4 \\ 1/3 & 2/3 \end{bmatrix}$ so $[I - zP] = \begin{bmatrix} 1 - \frac{3}{4}z & -\frac{1}{4}z \\ -\frac{1}{3}z & 1 - \frac{2}{3}z \end{bmatrix}$

$$\det[I - zP] = 1 - \frac{17}{12}z + \frac{5}{12}z^2$$



To find the first row of $[I - zP]^{-1}$, we put the input $1 - \frac{17}{12}z + \frac{5}{12}z^2$ at node (1).

n	input to (1)	signal at (1)	signal at (2)
0	1	1	0
1	$-\frac{17}{12}$	$-\frac{7}{12} + \frac{3}{4}(1) = -\frac{2}{3}$	$\frac{1}{4}$
2	$\frac{5}{12}$	$+\frac{5}{12} - \frac{3}{4}(\frac{2}{3}) + \frac{1}{3}(\frac{1}{4}) = 0$	$\frac{1}{4}(\frac{2}{3}) + (\frac{1}{4})(-\frac{2}{3}) = 0$
3	0	0	0
transforms	$1 - \frac{17}{12}z + \frac{5}{12}z^2$	$1 - \frac{2}{3}z$	$\frac{1}{4}z$

Hence,
$$[I - zP]^{-1} = \frac{1}{1 - \frac{17}{12}z + \frac{5}{12}z^2}\begin{bmatrix} 1 - \frac{2}{3}z & \frac{1}{4}z \\ \cdots & \cdots \end{bmatrix}.$$
The second row is computed by putting the input into (2).

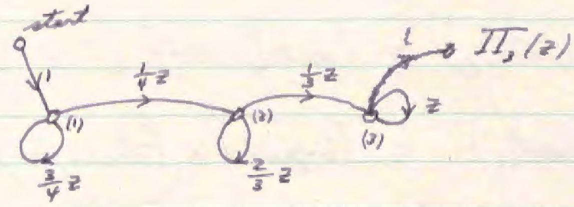
This method is good for complicated flow graphs, especially when interested in only one row.

First entrance time:

Up to now, we have studied only the state probabilities of the system. We now consider the problem: what is the probability that the system will occupy state i for the first time at time n?

Example: Taxi driver starts in (1), has probability of going to (2), will make trips in (2), but won't go back to (1) & instead goes home & stays there:

$$P = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} & 0 \\ 0 & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 1 \end{bmatrix}$$



We could calculate $\Pi_3(n)$, but we want $\pi_3^*(n)$, the probability that he will arrive home on the n^{th} move. π_3^* is the same as π_3 with the self-loop on (3) removed since $\pi_3(n) = \sum_{k=1}^{n} \pi_3^*(k)$.

$$\Pi_3^*(z) = \frac{\frac{1}{12}z^2}{(1-\frac{3}{4}z)(1-\frac{2}{3}z)}$$

This is no longer factorable by $\frac{1}{(1-z)}$ as our matrix is no longer stochastic; the last row is 0 0 0.

$$\Pi_3^*(z) = \frac{\frac{3}{4}z^2}{1-\frac{3}{4}z} + \frac{-\frac{2}{3}z^2}{1-\frac{2}{3}z}$$

$$\pi_3^*(n) = \frac{3}{4}\left(\frac{3}{4}\right)^{n-2} - \frac{2}{3}\left(\frac{2}{3}\right)^{n-2} = \left(\frac{3}{4}\right)^{n-1} - \left(\frac{2}{3}\right)^{n-1} \qquad n \geq 2.$$

Is he certain to arrive home eventually? He is if $\sum_{n=1}^{\infty} \pi_3^*(n) = 1$ that is, if $\Pi_3^*(1) = 1$

$$\bar{n} = \frac{d}{dz}\left[\Pi_3^*(z)\right]\Big|_{z=1} = 2 + \frac{d}{dz}\left[\frac{\Pi_3^*}{z^2}\right]\Big|_{z=1} = 2+5 = 7.$$

Finding mean of first entrance time from flow graph (a la Sittler):

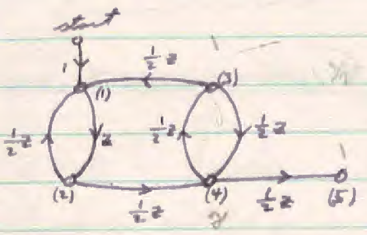
Given that we have a signal "1" at the end node corresponding to the fact that the system has fallen into the end state, we can count backward into the flow graph to see how many times the system moved before going into the final state. The average (or mean) value of the first entrance time is then just the sum of the transitions at each node:

Using the preceding example,

node	transitions
(3)	1
(2)	3
(1)	3 = 4−1

} We must subtract one from the first node since we attribute no delay to the first move into the system.

$\therefore \; \bar{n} = 1 + 3 + 3 = 7$

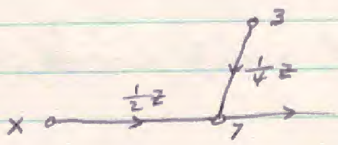


node	n
(5)	1
(4)	2
(3)	1
(2)	$2\left[2 - \frac{1}{2}(1)\right] = 3$
(1)	$3 - 1 = 2$

} $\bar{n} = 9$

Subtract off branches ~~leaving~~ entering the node with the signal we are working backwards from.

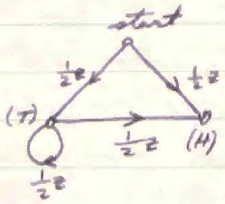
E.g.



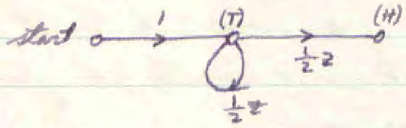
$x = 2\left[7 - \frac{1}{4}(3)\right]$

$\boxed{\text{Example}}$ A: obtain a head for the first time on the n^{th} toss of a fair coin.

$$P_r(A) = a(n)$$



But we don't care about the zeroth toss, so we can directly start the system off in (T) at $n=0$



$$A(z) = \frac{\frac{1}{2}z}{1-\frac{1}{2}z} \longrightarrow a(n) = \frac{1}{2}\left(\frac{1}{2}\right)^{n-1} = \underline{\left(\frac{1}{2}\right)^{n} \quad n \geq 1}$$

We can find $\bar{n}$ in ~~the~~ two ways; from the flow graph, or by differentiating $A(z)$.

From the flow graph, $\bar{n} = 1 + (2-1) = 1+1 = 2$.

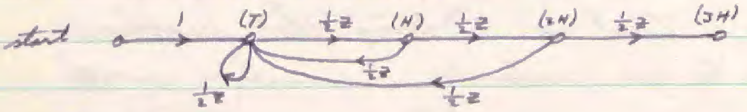
By differentiation, $\bar{n} = 1 + \frac{1}{2}\frac{d}{dz}\left(\frac{1}{1-\frac{1}{2}z}\right)\Big|_{z=1}$

$$\bar{n} = 1 + \frac{1}{2}\left(\frac{1}{1-\frac{1}{2}z}\right)^2\left(\frac{1}{2}\right) = 1 + \frac{1}{4}\left(\frac{1}{\frac{1}{4}}\right) = 1+1 = 2.$$

The variation σ_n^2 can not be found readily from the flow graph. Finding it by differentiation, we get

$$\sigma_n^2 = A''(1) + 2 - 2^2 = 4 + 2 - 4 = 2.$$

$\boxed{\text{Example}}$ B: the run HHH occurs for the first time on the n^{th} toss.
$Pr(B) = b(n)$.



$$B(z) = \frac{\frac{1}{8} z^3}{1 - \frac{1}{2} z - \frac{1}{4} z^2 - \frac{1}{8} z^3} = z^3 B^*(z)$$

$$\left.\frac{dB^*}{dz}\right|_1 = \frac{1}{8} \left.\frac{\left(\frac{1}{2} + \frac{1}{2} z + \frac{3}{8} z^2\right)}{[1 - (\ \ )]^2}\right|_1 = 8\left(\frac{11}{8}\right) = 11 = \overline{n^*}$$

$$\bar{n} = 11 + 3 = 14.$$

From the flow graph, $\bar{n} = 1 + 2 + 4 + (8-1) = 14$ ✓

<u>Finding mean & variance by going from z-transform to Laplace transform</u>:

Let $z = e^{-s}$; then expand $A(s)$ in a series $a_0 + a_1 s + a_2 s^2 + \dots = A(s)$. We need carry out only terms in s^2 to find $\bar{n}$ & $\overline{n^2}$ by differentiation as all higher order terms go to zero when we evaluate $A'(s)$ or $A''(s)$ at $s = 0$.

<u>Example on preceeding page</u>:

$$A(z) = \frac{\frac{1}{2} z}{1 - \frac{1}{2} z} = \frac{z}{2 - z} = \frac{e^{-s}}{2 - e^{-s}} = \frac{1 - s + \frac{s^2}{2} - \frac{s^3}{6} + \dots}{2 - (\cdot \ \cdot \ \cdot \ \cdot \ \cdot)}$$

$$A(s) = \frac{1 - s + \frac{s^2}{2} - \frac{s^3}{6} + \dots}{1 + s - \frac{s^2}{2} + \frac{s^3}{6} + \dots} = 1 - 2s + 3s^2 + \dots$$ by long division.

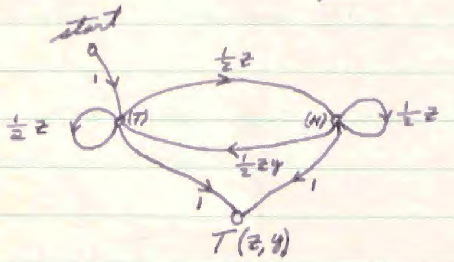
As a quick check, $A(0) = 1$ ✓.

$$\left.\begin{aligned} \bar{n} &= -\left.\frac{dA}{ds}\right|_{s=0} = 2 \ ✓ \\ \overline{n^2} &= \left.\frac{d^2A}{ds^2}\right|_{s=0} = 6 \ ✓ \end{aligned}\right\}$$ see p. 16

Tagging variables:

In some problems, we want to know the number of times a specific transition takes place. As a tool in solving this problem, we "tag" the branch corresponding to the particular transition with a tagging variable:

$\boxed{\text{Example}}$ Find the mean and variance of the number of times we get a (HT) sequence in n tosses.



y is our tagging variable corresponding to a H-T transition.

$$P(n,k) = \Pr\{k \text{ HT sequences in } n \text{ tosses}\}$$

$$\boxed{T(z,y) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} P(n,k)\, z^n y^k}$$

$$T(z,1) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} P(n,k)\, z^n = \sum_{n=0}^{\infty} z^n \quad \text{since} \quad \sum_{k=0}^{\infty} P(n,k) = 1$$

$$\therefore \boxed{T(z,1) = \frac{1}{1-z}} \quad \text{if} \quad \sum_{k=0}^{\infty} P(n,k) = 1$$

$$\left.\frac{\partial T}{\partial y}\right|_{y=1} = \left.\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} k\, P(n,k)\, z^n y^{k-1}\right|_{y=1} = \sum_n \Big[\sum_k k\, P(n,k)\Big] z^n = \sum_n \overline{k(n)}\, z^n$$

Hence, $$\boxed{\left.\frac{\partial T(z,y)}{\partial y}\right|_{y=1} = \sum_{n=0}^{\infty} \overline{k(n)}\, z^n \equiv \overline{K}(z)}$$

$$\frac{\partial^2 T}{\partial y^2} = \sum_k \sum_n P(n,k)\, z^n\, k(k-1)\, y^{k-2}$$

$$\left.\frac{\partial^2 T}{\partial y^2}\right|_{y=1} = \sum_n \Big[\sum_k k^2 P(n,k) - \sum_k k\, P(n,k)\Big] z^n = \sum_n \Big[\overline{k^2(n)} - \overline{k(n)}\Big] z^n$$

$$\boxed{\left.\frac{\partial^2 T(z,y)}{\partial y^2}\right|_{y=1} = \overline{K^2}(z) - \overline{K}(z)}$$ (c.f. p. 26)

For the problem used as an example,

$$T(z,y) = \frac{1}{1 - z + \frac{1}{4}z^2(1-y)} \quad ; \quad T(z,1) = \frac{1}{1-z}$$

$$\left.\frac{\partial T(z,y)}{\partial y}\right|_{y=1} = \frac{1/4\, z^2}{(1-z)^2} = \bar{K}(z) \longrightarrow \bar{K}(n) = \frac{1}{4}(n-1)$$

$$\left.\frac{\partial^2 T}{\partial y}\right|_{y=1} = \frac{\frac{1}{8} z^4}{(1-z)^3} = \bar{K^2}(z) - \bar{K}(z)$$

To evaluate the inverse transform, we need a new theorem.

$$F(z) = \sum_n f(n) z^n \Rightarrow F'(z) = \sum_n n f(n) z^{n-1}$$

Hence, $\boxed{zF'(z) = \sum_n n\, f(n)\, z^n}$

We can now extend our table of "singularity functions":

$f(n)$	$F(z)$
1	$\frac{1}{1-z}$
n	$\frac{z}{(1-z)^2}$
n^2	$\frac{z(z+1)}{(1-z)^3}$

For the transform of $\frac{z^4}{(1-z)^3}$, we try $n^2 - n$

$$n^2 - n \leftrightarrow \frac{z(z+1)}{(1-z)^3} - \frac{z(1-z)}{(1-z)^3} = \frac{2z^2}{(1-z)^3}$$

Hence, $\bar{K^2}(n) - \bar{K}(n) = \left(\frac{1}{8}\right)\left(\frac{1}{2}\right)\left[(n-2)^2 - (n-2)\right] = \frac{n^2 - 5n + 6}{16}$

$$\sigma_n^2 = \left[\bar{K^2}(n) - \bar{K}(n)\right] + \bar{K}(n) - \bar{K}(n)^2 = \frac{n^2-5n+6}{16} + \frac{n-1}{4} - \left(\frac{n-1}{4}\right)^2$$

$$\sigma_n^2 = \frac{n+1}{16}$$

Example, continued: Find $P(n,0)$

By inspection, this can happen as
$$\left.\begin{array}{c} TTT\cdots\cdots TT \\ TTT\cdots\cdots TH \\ TTT\cdots\cdots HH \\ \text{- - - - - - - -} \\ HHH\cdots\cdots HH \end{array}\right\} n+1 \text{ ways}$$

Each way is equally likely with probability $\left(\frac{1}{2}\right)^n$ so

$$P(n,0) = \frac{n+1}{2^n}$$

Using our transform methods, we get

$$\boxed{T(z,0) = \sum_n P(n,0)\, z^n \quad \text{since} \quad 0^k = \begin{cases} 1, & k=0 \\ 0, & k\neq 0 \end{cases}}$$

Applying this to our problem,

$$T(z,0) = \frac{1}{1-z+\frac{1}{4}z^2} = \frac{1}{(1-\frac{1}{2}z)^2}$$

$$\therefore P(n,0) = 2\left(\tfrac{1}{2}\right)^{n+1}(n+1) \quad \text{since} \quad \frac{\frac{1}{2}z}{(1-\frac{1}{2}z)^2} \Leftrightarrow \left(\tfrac{1}{2}\right)^n n$$

$$P(n,0) = \left(\tfrac{1}{2}\right)^n (n+1) = \frac{n+1}{2^n} \checkmark$$

Still another way to evaluate $P(n,k)$ for specific k ($\overset{n \text{ small}}{\cancel{k \text{ small}}}$) is to expand $T(z,y)$ in powers of y; the coefficient of y^k is the z transform of $P(n,k)$:

$$T(z,y) = \frac{1}{1-z+\frac{1}{4}z^2(1-y)} = \frac{1}{1-z\left[1-\frac{1}{4}z(1-y)\right]} = 1+z[\ \ ] + z^2[\ \ ]^2 + \cdots$$

$$T(z,y) = 1+z-\tfrac{1}{4}z^2+\tfrac{1}{4}z^2y+z^2+z^2-\cancel{\tfrac{1}{2}}z^3+\tfrac{1}{2}z^3y+\tfrac{1}{16}z^4-\tfrac{1}{8}z^4y+\tfrac{1}{8}z^4y^2+z^3+\cdots$$

good to terms in z^3.

$$T(z,y) = \underbrace{\left(1+z+\tfrac{3}{4}z^2+\tfrac{1}{2}z^3+\cdots\right)}_{P(n,0)} + y\underbrace{\left(\tfrac{1}{4}z^2+\tfrac{1}{2}z^3+\cdots\right)}_{P(n,1)} + y^2\underbrace{(\cdots\cdots)}_{P(n,2)} + \cdots$$

$P(1,0) = 1$ $P(1,1) = 0$

$P(2,0) = \frac{3}{4}$ $P(2,1) = \frac{1}{4}$ etc.

$P(3,0) = \frac{1}{2}$ $P(3,1) = \frac{1}{2}$

First recurrence time in Markov processes:

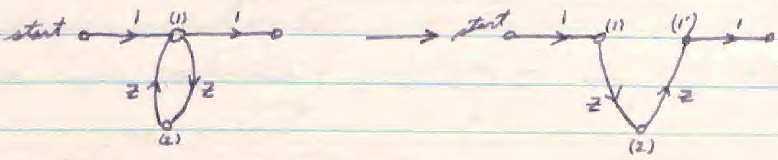
We want now to find the probability that the system returns to its starting state for the first time on the n^{th} transition.

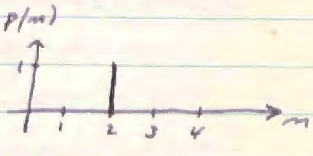
We are really considering a transient problem. We can find the first recurrence time by splitting the first node into two nodes, one containing all incoming branches and the other, all outgoing branches. We now just write the transmission "around the node" as in the first occurence time problems we have already considered.

$$\mathcal{P}(n) = \Pr\{\text{first recurrence occurs at time } n.\}$$

$$P(z) \equiv \sum_n \mathcal{P}(n)\, z^n$$

$\boxed{\text{Examples}}$

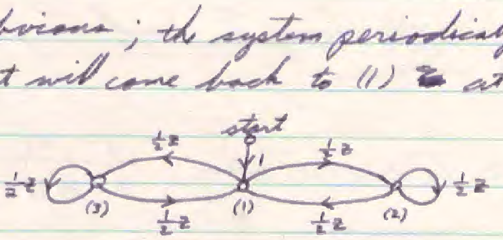




$$P(z) = z^2$$

This system is rather obvious; the system periodically jumps from (1) to (2), so if it is started in (1), it will come back to (1) at $n = 2$.

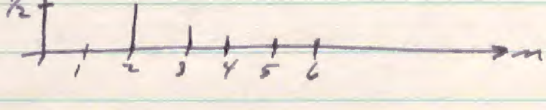
3-state problem:



Flip a coin to see which way we go $\left\{\begin{array}{l}T, \text{ we go left}\\ H, \text{ we go right}\end{array}\right\}$

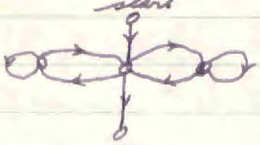
$$P(z) = \cancel{z\left(\tfrac{1}{2}z\right)\left(\tfrac{1}{1-\frac{1}{2}z}\right)\left(\tfrac{1}{2}z\right)} = \frac{\frac{1}{2}z^2}{1-\frac{1}{2}z} \qquad P(1) = 1 \checkmark$$

$$\mathcal{P}(n) = \tfrac{1}{2}\left(\tfrac{1}{2}\right)^{n-2} = \left(\tfrac{1}{2}\right)^{n-1} \qquad n \geq 2$$



$$\bar{n} = P'(z)\Big|_{z=1} = 2 + \tfrac{1}{2}(2) = 2 + 1 = 3.$$

Relation of first recurrence time mean to π: limiting steady state probability.



Regarding the split node transmission $P(z)$ as one branch, we can write

$$\boxed{\Pi(z) = \frac{1}{1-P(z)}}$$

For the example on the last page, $\Pi(z) = \dfrac{1-\frac{1}{2}z}{(1-z)(1+\frac{1}{2}z)} = \dfrac{1/3}{1-z} + \dfrac{2/3}{1-\frac{1}{2}z}$

$$\pi(n) = \frac{1}{3} + \frac{2}{3}\left(-\frac{1}{2}\right)^n \rightarrow \pi = 1/3 = \frac{1}{\bar{n}}.$$

Theorem:
$$\boxed{\left.\frac{dF(z)}{dz}\right|_{z=1} = \lim_{z\to 1} \frac{1-F(z)}{1-z} = \bar{n}}$$

Proof:
$$\boxed{\lim_{z\to 1} \frac{1-F(z)}{1-z} = \lim_{z\to 1} \frac{-F'(z)}{-1} = \lim_{z\to 1} F'(z) = F'(1) = \bar{n} \blacksquare}$$

This gives us a new way to find $\bar{n}$.

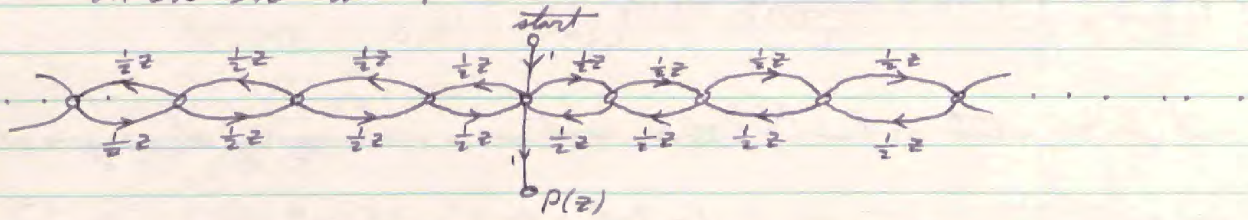
Now,
$$\pi = \lim_{z\to 1} (1-z)\,\Pi(z) = \lim_{z\to 1} \frac{1-z}{1-P(z)} = \frac{1}{\displaystyle\lim_{z\to 1} \frac{1-P(z)}{1-z}}$$

$$\boxed{\pi = \frac{1}{\left.\frac{dP(z)}{dz}\right|_{z=1}} = \frac{1}{\bar{n}}}$$

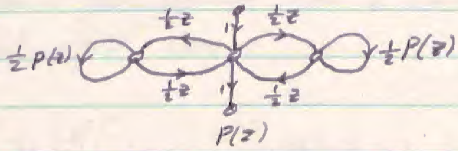
Random walk problem in one dimension:

$P(n) = Pr\{$ we get back to starting position on ~~first~~ n^{th} toss $\}$ ~~on it flip~~

If we were flipping a coin, $P(n)$ would be the probability that the ~~no~~ total number of heads equals the total numbers of tails for the first time on the n^{th} toss.



The total transmission is $P(z)$, so by symmetry, the transmission of each half of the flow graph is $\frac{1}{2}P(z)$. But since the $\frac{1}{2}$ chain is infinitely long in one direction, ~~so~~ the transmission through any node is $\frac{1}{2}P(z)$. This reduces the problem to:



Remember to split the center node as this is a first recurrence time problem.

$$P(z) = 2\left[\frac{\frac{1}{4}z^2}{1-\frac{1}{2}P(z)}\right] = \frac{\frac{1}{2}z^2}{1-\frac{1}{2}P(z)}$$

or $\quad P^2(z) - 2P(z) + z^2 = 0$

$$P(z) = \frac{1}{2}\left[2 \pm \sqrt{4-4z^2}\right] = 1 \pm \sqrt{1-z^2}$$

$P(0) = 0$ for an acceptable distribution (can't return on zeroth move)

so $\quad \boxed{P(z) = 1 - \sqrt{1-z^2}}$

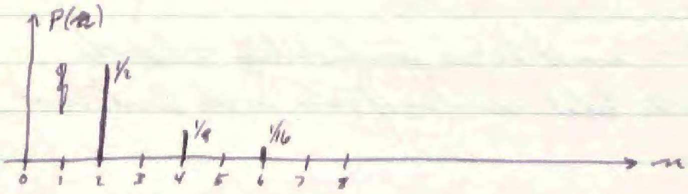
$P(1) = 1$ so we will eventually return.

$P'(z) = \frac{z}{\sqrt{1-z^2}}$ so $P'(1) = \infty$ & $\pi = 0$ & $\bar{n} = \infty$

so it will take us an average of infinitely many moves to return.

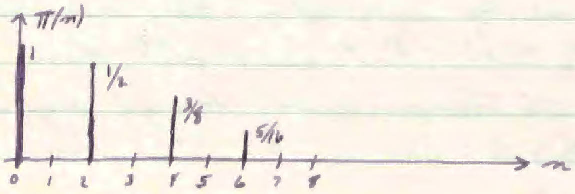
<u>Random walk, continued</u>:

$$P(z) = 1 - \sqrt{1-z^2} = 1 - \left(1 - \tfrac{1}{2}z^2 - \tfrac{1}{8}z^4 - \dots\right) = \tfrac{1}{2}z^2 + \tfrac{1}{8}z^4 + \tfrac{1}{16}z^6 + \dots$$



Now we ask for the state probability, $\pi(n)$, that we are occupying the origin at time n:

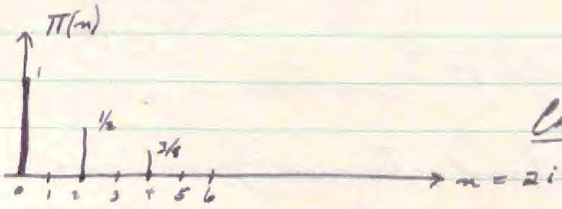
$$\Pi(z) = \frac{1}{1-P(z)} = \frac{1}{\sqrt{1-z^2}} = 1 + \tfrac{1}{2}z^2 + \tfrac{3}{8}z^4 + \tfrac{5}{16}z^6 + \dots$$



By physical reasoning, we can write

$$\pi(2i) = \binom{2i}{i}\left(\frac{1}{2}\right)^i\left(\frac{1}{2}\right)^i = \cancel{\binom{2i}{i}\left(\frac{1}{2}\right)^{2i}} = \binom{2i}{i}\left(\frac{1}{2}\right)^{2i}$$

$$= \frac{(2i)!}{(i!)^2}\left(\frac{1}{2}\right)^{2i}$$



<u>Checks with above</u>

Congestion problems, or queueing theory:

Consider a system whose arrivals are Poisson distributed with mean λ (c.f. p. 18):

$$P_0(t) = P_r\{\text{no arrivals in time } t\}$$

$$P_0(t+dt) = P_0(t)[1-\lambda dt] \rightarrow \frac{P_0(t+dt)-P_0(t)}{dt} = -\lambda = \frac{dP_0}{dt}$$

$$\underline{P_0 = e^{-\lambda t}} \qquad \text{since } P_0(0) = 1.$$

The description of the input we are really interested in is the interval between arrivals $P(t)$.

$$P(t)\,dt = P_0(t)\,\lambda dt \longrightarrow \boxed{P(t) = \lambda e^{-\lambda t}}$$

Suppose the system has a service time which is exponentially distributed with mean μ:

$\boxed{\text{1 unit is served at a time}}$

$$\lambda e^{-\lambda t} \quad \circ\!\xrightarrow{\mu e^{-\mu t}}\!\circ$$

Define $P_n(t) \equiv P_r\{n$ units are in the system at time t; either being served (1) or waiting $(n-1)\}$

$$P_n(t+dt) = P_{n+1}(t)\mu dt + P_n(t)[1-\lambda dt][1-\mu dt] + P_{n-1}(t)\lambda dt \qquad n \geq 1$$

$$= P_{n+1}(t)\mu dt + P_n(t)[1-(\lambda+\mu)dt] + P_{n-1}(t)\lambda dt$$

$$\boxed{\frac{dP_n(t)}{dt} = \mu P_{n+1}(t) - (\lambda+\mu)P_n(t) + \lambda P_{n-1}(t) \qquad n \geq 1}$$

For $n = 0$, $\quad P_0(t+dt) = P_1(t)\mu dt + P_0(t)[1-\lambda dt]$

$$\boxed{\frac{dP_0(t)}{dt} = \mu P_1(t) - \lambda P_0(t) \qquad n = 0}$$

This problem is too hairy to solve, so we look for a <u>steady state solution</u>:

If the system is in steady state, $0 = \frac{dP_n(t)}{dt} = \frac{dP_0(t)}{dt}$

and our equations reduce to a set of linear difference equations with constant coefficients:

$$\boxed{\begin{aligned} &\mu P_{n+1} - (\lambda+\mu) P_n + \lambda P_{n-1} = 0 \quad , n \geq 1 \\ &\mu P_1 - \lambda P_0 = 0 \quad , n = 0 \end{aligned}}$$

We now try to solve these equations by the z-transform methods we have developed.

> To do this, our equations must be written so as to be valid over the range $n \geq 0$

$$\boxed{\begin{aligned} &\mu P_{n+2} - (\lambda+\mu) P_{n+1} + \lambda P_n = 0 \quad , \quad n \geq 0 \\ &\mu P_1 - \lambda P_0 = 0 \quad , \quad n = 0 \end{aligned}}$$

Now let $P(z) = \sum_{n=0}^{\infty} P_n z^n$

$$\mu\left[z^{-2} P(z) - P_0 z^{-2} - P_1 z^{-1}\right] - (\lambda+\mu)\left[z^{-1} P(z) - P_0 z^{-1}\right] + \lambda P(z) = 0$$

$$\left[\lambda z^2 - (\lambda+\mu) z + \mu\right] P(z) = \mu P_0 (1-z) + (\mu P_1 - \lambda P_0) z = \mu P_0 (1-z)$$

since $\mu P_1 - \lambda P_0 = 0$

$$P(z) = \frac{\mu P_0 (1-z)}{\mu - (\lambda+\mu) z + \lambda z^2} = \frac{P_0 (1-z)}{1 - (1+\rho) z + \rho z^2} = \frac{P_0 (1-z)}{(1-z)(1-\rho z)}$$

$P(z) = \frac{P_0}{1-\rho z}$ but $P(1) = 1 = \frac{P_0}{1-\rho}$ or $P_0 = 1-\rho$

$$\boxed{P(z) = \frac{1-\rho}{1-\rho z}} \qquad \boxed{\rho \equiv \frac{\lambda}{\mu} = \text{utilization factor}} \; \boxed{\rho < 1}$$

$$\boxed{P_n = (1-\rho)\rho^n} \ , \quad n \geq 0$$

$$\boxed{P_n = P_r\{n \text{ people in system}\} = P_r\{n-1 \text{ people waiting}\}}$$

$$\bar{n} = \left.\frac{dP(z)}{dz}\right|_{z=1} = \frac{\rho}{1-\rho}$$

$$\sigma_n^2 = P''(1) + P'(1) - [P'(1)]^2 = \frac{\rho}{(1-\rho)^2}$$

$$\boxed{\text{Average wait} = \text{avg service time} \times \text{avg number of people} = \frac{1}{\mu}\bar{n} = \frac{\rho}{1-\rho}\frac{1}{\mu}}$$

<u>Simple transient problem</u>:

One machine, one repair system:
Probability of breakdown in dt is λdt.
Probability of repair in dt is μdt.

$P_0(t)$ = probability the machine is working at time t.
$P_1(t)$ = " " " " not " " " ".

$$P_0(t+dt) = P_0(t)[1-\lambda dt] + P_1(t)\mu dt$$
$$P_1(t+dt) = P_0(t)\lambda dt + P_1(t)[1-\mu dt]$$

$$\left\{\begin{aligned} \frac{dP_0(t)}{dt} &= -\lambda P_0(t) + \mu P_1(t) \\ \frac{dP_1(t)}{dt} &= \lambda P_0(t) - \mu P_1(t) \end{aligned}\right\}$$

Using Laplace transforms to solve these equations,

$$\left.\begin{aligned} sP_0(s) - P_0 &= -\lambda P_0(s) + \mu P_1(s) \\ sP_1(s) - P_1 &= \lambda P_0(s) - \mu P_1(s) \end{aligned}\right\} \quad \boxed{\begin{aligned} (s+\lambda)P_0(s) - \mu P_1(s) &= P_0 \\ \lambda P_0(s) + (s+\mu)P_1(s) &= P_1 \end{aligned}}$$

$$\Delta \equiv \det = s^2 + \lambda s + \mu s = s(s+\lambda+\mu)$$

$$P_0(s) = \frac{P_0(s+\mu) + \mu P_1}{\Delta} = \frac{P_0 s + \mu}{s(s+\lambda+\mu)}$$

$$P_1(s) = \frac{P_1(s+\lambda) + \lambda P_0}{\Delta} = \frac{P_1 s + \lambda}{s(s+\lambda+\mu)}$$

$$P_0(\infty) = \lim_{s\to 0} s P_0(s) = \frac{\mu}{\lambda+\mu}$$

$$P_1(\infty) = \lim_{s\to 0} s P_1(s) = \frac{\lambda}{\lambda+\mu}$$

$$P_0(s) = \frac{\left(\frac{\mu}{\lambda+\mu}\right)}{s} + \frac{P_0 - \frac{\mu}{\lambda+\mu}}{s+(\lambda+\mu)}$$

$$P_1(s) = \frac{\left(\frac{\lambda}{\lambda+\mu}\right)}{s} + \frac{P_1 - \frac{\lambda}{\lambda+\mu}}{s+(\lambda+\mu)}$$

$$\boxed{\begin{aligned} P_0(t) &= \frac{\mu}{\lambda+\mu} + \left(P_0 - \frac{\mu}{\lambda+\mu}\right)e^{-(\lambda+\mu)t} = \frac{\mu}{\lambda+\mu}\left[1-e^{-(\lambda+\mu)t}\right] + P_0 e^{-(\lambda+\mu)t} \\ P_1(t) &= \frac{\lambda}{\lambda+\mu} + \left(P_1 - \frac{\lambda}{\lambda+\mu}\right)e^{-(\lambda+\mu)t} = \frac{\lambda}{\lambda+\mu}\left[1-e^{-(\lambda+\mu)t}\right] + P_1 e^{-(\lambda+\mu)t} \end{aligned}}$$

Now suppose we say the machine is definitely working at $t=0$:

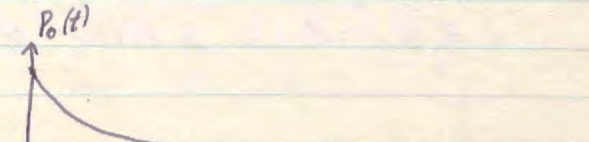
$$P_0 = 1 \quad , \quad P_1 = 0$$

$$P_0(t) = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t} = \frac{\mu}{\lambda+\mu}\left[1 + \rho e^{-(\lambda+\mu)t}\right]$$

$$P_1(t) = \frac{\lambda}{\lambda+\mu} - \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t} = \frac{\lambda}{\lambda+\mu}\left[1 - e^{-(\lambda+\mu)t}\right]$$

c.f. p. 99

System time constant $= (\lambda+\mu)$



Double transforms: $P(z,s) \leftrightarrow p(n,t)$:

Consider a distribution such that

$$p(n,t) = Pr\{n \text{ arrivals in time } t \text{ if } Pr(\text{arrival in } dt) = \lambda dt\}$$

As found previously, the differential equations relating the p's are:

$$\frac{dp(n+1,t)}{dt} = \lambda p(n,t) - \lambda p(n+1,t) \quad , n \geq 0$$

$$\frac{dp(0,t)}{dt} = -\lambda p(0,t) \quad , n = 0$$

Now, we define the transforms:

$$\boxed{\begin{aligned} P(z,s) &\equiv \sum_{n=0}^{\infty} \int_0^{\infty} p(n,t)\, z^n e^{-st}\, dt \\ G(n,s) &\equiv \int_0^{\infty} p(n,t)\, e^{-st}\, dt \\ H(z,t) &\equiv \sum_{n=0}^{\infty} p(n,t)\, z^n \end{aligned}}$$

Call $P(z,s) = T_r\{p(n,t)\}$

Note that $P(z,s) = \sum_{n=0}^{\infty} G(n,s)\, z^n = \int_0^{\infty} H(z,t)\, e^{-st}\, dt$

$$p(n+1,t) \leftrightarrow \sum_{n=0}^{\infty} \int_0^{\infty} p(n+1,t)\, z^n e^{-st}\, dt = \sum_{j=1}^{\infty} \int_0^{\infty} p(j,t)\, z^{j-1} e^{-st}\, dt$$

$$\leftrightarrow z^{-1}\left[P(z,s) - \int_0^{\infty} p(0,t)\, e^{-st}\, dt\right]$$

so $\boxed{p(n+1,t) \leftrightarrow z^{-1}\left[P(z,s) - G(0,s)\right]}$

$$\frac{dp(n,t)}{dt} \leftrightarrow \sum_{n=0}^{\infty} \int_0^{\infty} \frac{dp(n,t)}{dt} z^n e^{-st}\, dt = \sum_{n=0}^{\infty} z^n \left[s\, G(n,s) - p(n,0)\right]$$

$$\boxed{\frac{dp(n,t)}{dt} \leftrightarrow s\, P(z,s) - H(z,0)}$$

$$\frac{dP(n+1,t)}{dt} \leftrightarrow sz^{-1}\Big[P(z,s)-G(0,s)\Big] - z^{-1}\Big[H(z,0)-P(0,0)\Big]$$

$$\leftrightarrow \lambda P(z,s) - \lambda z^{-1}\big[P(z,s)-G(0,s)\big]$$ from equation for $n \geq 0$

The equation for $n=0$ gives $sG(0,s) - P(0,0) = -\lambda G(0,s)$

so $$P(z,s)\big[s+\lambda-\lambda z\big] = G(0,s)\big[s+\lambda\big] + H(z,0) - P(0,0)$$

$$G(0,s) = \frac{P(0,0)}{s+\lambda}$$

so $$\boxed{P(z,s) = \frac{H(z,0)}{s+\lambda-\lambda z}}$$

As an initial condition, let $P(0,0)=1$, $P(n,0)=0$, $n \geq 1$
then
$$H(z,0) = 1 \quad ; \quad P(1,s) = \frac{1}{s}$$

$$P(z,s) = \frac{1}{s+\lambda-\lambda z} = \frac{\frac{1}{s+\lambda}}{1-\frac{\lambda}{s+\lambda}z}$$

Inverse transforming, we get

$$G(n,s) = \frac{1}{s+\lambda}\left(\frac{\lambda}{s+\lambda}\right)^n = \frac{\lambda^n}{(s+\lambda)^{n+1}}$$

Now $$\frac{1}{s+\lambda} = \mathcal{L}\left\{e^{-\lambda t}\right\} \quad \& \quad \frac{dF(s)}{ds} = -\mathcal{L}\left\{t\,f(t)\right\}$$

so $$\frac{1}{(s+\lambda)^2} = \mathcal{L}\left\{te^{-\lambda t}\right\} \quad ; \quad \frac{2}{(s+\lambda)^3} = \mathcal{L}\left\{t^2e^{-\lambda t}\right\}$$ etc.

and

$$\frac{1}{(s+\lambda)^n} = \mathcal{L}\left\{\frac{t^n}{n!}e^{-\lambda t}\right\}$$

Hence, we have the relation

$$\boxed{P(n,t) = \lambda^n \frac{t^n}{n!} e^{-\lambda t} = \frac{(\lambda t)^n}{n!} e^{-\lambda t}}$$

This is the same distribution we found previously (cf. p. 18 + p. 27-28)

Another way of getting this answer is to transform first to time:

$$H(z,t) = e^{-(\lambda - \lambda z)t} = e^{-\lambda t} e^{\lambda z t}$$

$$= e^{-\lambda t}\left[1 + (\lambda t) z + \frac{(\lambda t)^2}{2!} z^2 + \frac{(\lambda t)^3}{3!} z^3 + \dots \right]$$

and again,

$$P(n,t) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}.$$

<u>Waiting-line problem</u>: $\left\{ \begin{array}{l} \text{Poisson distributed arrivals} \\ \text{Exponentially distributed service times} \end{array} \right\}$

Consider a single channel; steady state; allow ∞ queues.

$$P(n,t) = Pr\{n \text{ customers are discharged in time interval } t\}$$

$$P(j) = Pr\{j \text{ units in the system at } t = 0\}$$

From the past examination of queues (p 65), we know that

$$P(j) = (1-\rho)\rho^j \quad , \rho = \frac{\lambda}{\mu}$$

Now, $P(n,t/j) = Pr\{n$ units are discharged in time t, given that there were j units in the system at $t = 0\}$.

So,

$$\boxed{P(n,t) = \sum_{j=0}^{\infty} P(n,t/j)\,P(j) = \sum_{j=0}^{\infty} P(n,t/j)\,(1-\rho)\rho^j}$$

For $n \geq 1$

$$P(n, t+dt \mid j) = \mu\, dt\, P(n-1, t \mid j-1) + \lambda\, dt\, P(n, t \mid j+1) + (1-\lambda dt - \lambda\mu dt)\, P(n, t \mid j)$$

discharge   arrival

Now, drop the t notation

$j \geq 1$, $\quad \dfrac{dP(n|j)}{dt} = \mu P(n-1|j-1) + \lambda P(n|j+1) - (\lambda+\mu) P(n|j)$

$j = 0$, $\quad \dfrac{dP(n|0)}{dt} = \lambda P(n|1) - \lambda P(n|0)$ $\quad$ (no discharge allowed)

For $n=0$, $j \geq 1$, $\quad \dfrac{dP(0|j)}{dt} = \lambda P(0|j+1) - (\lambda+\mu) P(0|j)$

$j = 0$, $\quad \dfrac{dP(0,0)}{dt} = \lambda P(0|1) - \lambda P(0,0)$

In our notation with t omitted,

$$\boxed{P(n) = \sum_{j=0}^{\infty} P(n|j)(1-\rho)\rho^{j}}$$

We now try to compute this summation:

$n, j \geq 1$; $\quad \displaystyle\sum_{j=1}^{\infty} \frac{dP(n|j)}{dt}(1-\rho)\rho^{j} = \#$

$$= \sum_{j=1}^{\infty} \mu P(n-1|j-1)(1-\rho)\rho^{j} + \sum_{j=1}^{\infty} \lambda P(n|j+1)(1-\rho)\rho^{j} - (\lambda+\mu)\sum_{j=1}^{\infty} P(n|j)(1-\rho)\rho^{j}$$

$n \geq 1$, $j = 0$; $\# \quad \dfrac{dP(n,0)}{dt}(1-\rho) = \lambda(1-\rho)P(n|1) - \lambda(1-\rho)P(n|0)$

$$\sum_{j=0}^{\infty} \frac{dP(n|j)}{dt}(1-\rho)\rho^{j} = \sum_{j=1}^{\infty} \mu P(n-1|j-1)(1-\rho)\rho^{j} + \sum_{j=0}^{\infty} \lambda P(n|j+1)(1-\rho)\rho^{j} - \lambda \sum_{j=0}^{\infty} P(n|j)(1-\rho)\rho^{j} - \mu \sum_{j=1}^{\infty} P(n|j)(1-\rho)\rho^{j}$$

$$\sum_{j=1}^{\infty} P(n-1|j-1)(1-\rho)\rho^{j} = \sum_{k=0}^{\infty} P(n-1|k)(1-\rho)\rho^{k+1} = \rho P(n-1)$$

$$\sum_{j=0}^{\infty} P(n|j+1)(1-\rho)\rho^{j} = \sum_{k=1}^{\infty} P(n|k)(1-\rho)\rho^{k-1} = \frac{1}{\rho}\left[P(n) - (1-\rho)P(n|0)\right]$$

$$\sum_{j=1}^{\infty} P(n|j)(1-\rho)\rho^{j} = P(n) - (1-\rho)P(n|0)$$

$$\frac{dP(n)}{dt} = \mu\rho P(n-1) + \frac{\lambda}{\rho}\left[P(n) - (1-\rho)P(n|0)\right] - \lambda P(n) - \mu\left[P(n) - (1-\rho)P(n|0)\right]$$

$$= \lambda P(n-1) - \lambda P(n)$$

$$\to \quad \boxed{\frac{dP(n,t)}{dt} = -\lambda P(n,t) + \lambda P(n-1,t) \quad , n \geq 1}$$

For $n=0$; $j \geq 1$: $\sum_{j=1}^{\infty} \frac{dP(0|j)}{dt}(1-\rho)\rho^j = \sum_{j=1}^{\infty} \lambda P(0|j+1)(1-\rho)\rho^j - (\lambda+\mu)\sum_{j=1}^{\infty} P(0|j)(1-\rho)\rho^j$

$j=0$: $\frac{dP(0|0)}{dt}(1-\rho) = \lambda P(0|1)(1-\rho) - \lambda P(0|0)(1-\rho)$

$$\sum_{j=0}^{\infty} \frac{dP(0|j)}{dt}(1-\rho)\rho^j = \sum_{j=0}^{\infty} \lambda P(0|j+1)(1-\rho)\rho^j - \lambda \sum_{j=0}^{\infty} P(0|j)(1-\rho)\rho^j - \mu \sum_{j=1}^{\infty} P(0|j)(1-\rho)\rho^j$$

$$\frac{dP(0)}{dt} = \frac{\lambda}{\rho}\left[P(0) - (1-\rho)P(0|0)\right] - \lambda P(0) - \mu\left[P(0) - (1-\rho)P(0|0)\right]$$

$$\boxed{\frac{dP(0,t)}{dt} = -\lambda P(0,t) \quad , n=0}$$

If ~~P(0,0)~~ $P(0,0) = 1$, the solution is

$$P(n,t) = \frac{(\lambda t)^n}{n!} e^{-\lambda t} \Bigg\} \text{ same distribution as arrival.}$$

$P_r\{\text{output in } dt\} = P_r\{\text{channel not empty}\} \times P_r\{\text{service in } dt\}$

$= \rho\mu\, dt = \lambda dt$; assuming independence.

Car-wash problem:

The system can accept a car every T seconds in time. Assume we know

$$\varphi_K \equiv P_r\{K \text{ arrivals in time } T\} \text{ , mean } < 1$$

We want to find

$$P_K \equiv P_r\{K \text{ units in the system}\}$$

We consider only one person serviced at a time, so $K = 1 + \#$ in queue.

Now, $$P_K = P_{K+1}\varphi_0 + P_K\varphi_1 + P_{K-1}\varphi_2 + \cdots + P_1\varphi_K + P_0\varphi_K$$

(Note that the one person (if any) who was being served previously is now gone — this explains why we have both $P_1\varphi_K$ and $P_0\varphi_K$.)

Define:

$$\left.\begin{aligned} \Phi(z) &= \sum_{K=0}^{\infty}\varphi_K z^K \\ P(z) &= \sum_{K=0}^{\infty} P_K z^K \end{aligned}\right\} \quad \underline{\Phi(1) = 1 \ ; \ \Phi'(1) = \mu \ ; \ \Phi''(1) = \sigma^2 + \mu^2 - \mu.}$$

We can now write down the equations for P_K for all values of K.

$$\begin{aligned} (0) \quad P_0 &= P_0\varphi_0 + P_1\varphi_0 \\ (1) \quad P_1 &= P_0\varphi_1 + P_1\varphi_1 + P_2\varphi_0 \\ (2) \quad P_2 &= P_0\varphi_2 + P_1\varphi_2 + P_2\varphi_1 + P_3\varphi_0 \\ &\cdots \end{aligned}$$

If we multiply the n^{th} equation by z^K and sum over all P_n, we get

$$P(z) = P_0\Phi(z) + P_1\Phi(z) + zP_2\Phi(z) + z^2P_3\Phi(z) + \cdots$$

$$= \Phi(z)\left[P_0 + P_1 + P_2 z + P_3 z^2 + \cdots\right] = \Phi(z)\frac{1}{z}\left[-P_0 + P_0 z + P_0 + P_1 z + P_2 z^2 + \cdots\right]$$

$$= \Phi(z)\left[\frac{(z-1)P_0 + P(z)}{z}\right]$$

Hence, $\quad zP(z) - \Phi(z)P(z) = (z-1)\,P_0\,\Phi(z)$

or, $\quad \boxed{P(z) = \dfrac{(z-1)P_0\,\Phi(z)}{z - \Phi(z)}}$

Now, we want to get rid of P_0, so we set $P_0 = P(1)$

$$P(1) = \lim_{z\to1} P(z) = \lim_{z\to1} \frac{P_0\left[(z-1)\Phi'(z) + \Phi(z)\right]}{1-\Phi'(z)} = \cancel{\lim_{z\to1} \frac{\Phi(z)}{1-\Phi'(z)}}$$

$$= \lim_{z\to1} \frac{P_0\,\Phi(z)}{1-\Phi'(z)} = \frac{P_0}{1-u} = 1 \quad \& \quad \boxed{P_0 = 1-u}$$

Hence, $\quad \boxed{P(z) = \dfrac{(1-u)(z-1)\,\Phi(z)}{z - \Phi(z)}}$

Now, we want to find out just what is the mean? ~~z~~

$P'(1)$ = average number of people in the system at the end of an interval T, when service is available.

~~But P'(1) is inde~~

But $P'(1)$ is indeterminate, as is the first application of L'Hopitals rule. So we try a new technique:

Take the log of $P(z)$:

$$\ln P(z) = \ln(1-u) + \ln(z-1) + \ln\Phi(z) - \ln\left[z - \Phi(z)\right]$$

Taking the derivative, we have

$$\frac{P'(z)}{P(z)} = \frac{1}{z-1} + \frac{\Phi'(z)}{\Phi(z)} - \frac{1-\Phi'(z)}{z-\Phi(z)}$$

$$P'(1) = u + \lim_{z\to1}\left[\frac{1}{z-1} - \frac{1-\Phi'(z)}{z-\Phi(z)}\right] = u + \lim_{z\to1}\left[\frac{1-\Phi(z) + (z-1)\Phi'(z)}{(z-1)(z-\Phi(z))}\right]$$

The limit is indeterminate, and so is the first application of L'Hopital's rule. After the second application, we get, however,

$$P'(1) = \mu + \frac{\Phi''(1)}{2[1-\Phi'(1)]} = \mu + \frac{\sigma^2 + \mu^2 - \mu}{2(1-\mu)}$$

or $$\boxed{P'(1) = \frac{1}{2}\left(\mu + \frac{\sigma^2}{1-\mu}\right)}$$

The advantage of taking the log was to simplify the derivatives. While they are bad as is, they are even worse if taken directly.

<u>Another approach to taking the limit of $P(z)$, $z \to 1$</u> :

Let $z = 1+\epsilon$; then $P'(z) = \frac{d}{d\epsilon}\left[P(1+\epsilon)\right]$

$$\Phi(1+\epsilon) = 1 + \epsilon\,\Phi'(1) + \frac{\epsilon^2}{2}\,\Phi''(1) + \cdots = 1 + \epsilon\mu + \frac{\epsilon^2}{2}(\sigma^2+\mu^2-\mu) + \cdots$$

$$P(z) = \frac{(1-\mu)(\epsilon)\left[1+\epsilon\mu+\frac{\epsilon^2}{2}(\sigma^2+\mu^2-\mu)+\cdots\right]}{1+\epsilon-\left[1+\epsilon\mu+\frac{\epsilon^2}{2}(\sigma^2+\mu^2-\mu)+\cdots\right]} = \frac{(1+\mu)\left[1+\epsilon\mu+\frac{\epsilon^2}{2}(\sigma^2+\mu^2-\mu)+\cdots\right]}{1-\mu-\frac{\epsilon^2}{2}(\sigma^2+\mu^2-\mu)+\cdots}$$

$$1 = \frac{d}{d\epsilon}P(1+\epsilon)\Big|_{\epsilon=0} = \mu + \frac{\sigma^2+\mu^2-\mu}{2(1-\mu)} = \frac{1}{2}\left(\mu + \frac{\sigma^2}{1-\mu}\right)$$

Let $\varphi_K = (1-\rho)\rho^K$, $K \geq 0$.

$$\Phi(z) = \frac{1-\rho}{1-\rho z} \quad ; \quad \bar{K} = \frac{\rho}{1-\rho} \quad ; \quad \sigma^2 = \frac{\rho}{(1-\rho)^2}$$

For stability, ~~ρ and $\bar{K}$ must be~~

$\bar{K} < 1$ or $\rho < \frac{1}{2}$ since $\bar{K}$ = # <u>arriving</u>.

The average number of people in the system is $\bar{n} = \frac{1}{2}\left(\mu + \frac{\sigma^2}{1-\mu}\right)$

Here,
$$\bar{n} = \frac{1}{2}\left[\frac{\rho}{1-\rho} + \frac{\frac{\rho}{(1-\rho)^2}}{1-\frac{\rho}{1-\rho}}\right] = \frac{\rho}{1-2\rho}$$

$\boxed{\text{As } \rho \to 1/2,\ \bar{n} \to \infty}$

Birth & death process:

$b_n \equiv$ birth rate at ~~time~~ n
$d_n \equiv$ death rate at ~~time~~ n

$P(n,t) = Pr\{n \text{ units in system at time } t\}$

$$P(n,t+dt) = P(n-1,t)\,b_{n-1}\,dt + P(n+1,t)\,d_{n+1}\,dt + P(n,t)\left[1-(d_n+b_n)dt\right]$$

$$\left\{\begin{array}{ll} \frac{d}{dt}P(n,t) = b_{n-1}P(n-1,t) - (b_n+d_n)P(n,t) + d_{n+1}P(n+1,t) & n \geq 1 \\ \frac{d}{dt}P(0,t) = -b_0 P(0,t) + d_1 P(1,t) & n = 0 \end{array}\right\}$$

Note that d_0 must be identically zero : $d_0 \equiv 0$.

If we have a steady state, we can write $\frac{d}{dt} = 0$ &

$$\boxed{\begin{array}{l} d_{n+2}P_{n+2} - (b_{n+1}+d_{n+1})P_{n+1} + b_n P_n = 0 \quad , n \geq 0 \\ -b_0 P_0 + d_1 P_0 \end{array}}$$

We note that subscripts always agree in multiplicative terms, so we define the new variables:

$$f(n) = d_n P_n \longleftrightarrow F(z) = \sum_{n=0}^{\infty} d_n P_n z^n = \sum_{n=0}^{\infty} f(n) z^n$$
$$g(n) = b_n P_n \qquad G(z) = \sum_{n=0}^{\infty} b_n P_n z^n = \sum_{n=0}^{\infty} g(n) z^n$$

$$f(n+k) \leftrightarrow \sum_{n=0}^{\infty} f(n+k) z^n = z^{-k}\left[F(z) - f(0) - z f(1) - \cdots - f(k-1) z^{k-1}\right]$$

Applying this to the above difference equation, we get

$$z^{-2}\left[F(z) - \cancel{d_0}P_0 - \cancel{d_1}P_1 z\right] - z^{-1}\left[G(z) - \cancel{b_0}P_0 + F(z) - \cancel{d_0}P_0\right] + G(z) = 0$$

$$(1-z)F(z) + (z^2-z)G(z) = 0$$

Hence, $\boxed{F(z) = z\,G(z)} \leftrightarrow \boxed{d_n P_n = b_{n-1} P_{n-1}} \quad n \geq 1$

In particular, consider the infinite numbers of equations:

$$[P_0 \;\; P_1 \;\; P_2 \;\ldots\;] \begin{bmatrix} -b_0 & b_0 & 0 & 0 & 0 & 0 & 0 \cdots \\ d_1 & -b_1-d_1 & b_1 & 0 & 0 & 0 & 0 \cdots \\ 0 & d_2 & -b_2-d_2 & b_2 & 0 & 0 & 0 \cdots \\ 0 & 0 & d_3 & -b_3-d_3 & b_3 & 0 & 0 \cdots \\ \cdot & \cdot & - & - & - & - & - \end{bmatrix}$$

$$P_n = \frac{b_{n-1}}{d_n} P_{n-1} \quad \rightarrow \quad P_1 = \frac{b_0}{d_1} P_0$$

$$P_2 = \frac{b_1}{d_2} P_1 = \frac{b_1 b_0}{d_2 d_1} P_0$$

$$\boxed{P_n = \prod_{i=1}^{n} \frac{b_{i-1}}{d_i} P_0}$$

<u>Example</u>: $\boxed{\text{Single waiting line}}$

Poisson arrivals $(\lambda) \quad \leftrightarrow \quad b_n = \lambda, \; n \geq 0$
Exponential service times $(\mu) \leftrightarrow d_n = \mu, \; n \geq 1$

$$P_n = P_0 \prod_{i=1}^{n} \frac{\lambda}{\mu} = P_0 \left(\frac{\lambda}{\mu}\right)^n, \quad n \geq 1$$

To evaluate P_0, we ~~may~~ use the requirement that $\sum_{n=0}^{\infty} P_n = 1$

$$1 = P_0 \sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n = P_0 \frac{1}{1-\frac{\lambda}{\mu}} \quad \Rightarrow \quad P_0 = 1 - \frac{\lambda}{\mu}$$

$$\boxed{P_n = \left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n = (1-\rho)\rho^n, \quad n \geq 0}$$

Example: 2 machines, how often are they broken down?

State 0 : both working – none broken
1 : one broken
2 : 2 broken

Assume breakdowns at Poisson distributed (λ)
service is exponentially distributed (μ)

$$A = \begin{bmatrix} -2\lambda & 2\lambda & 0 \\ \mu & -(\lambda+\mu) & \lambda \\ 0 & 2\mu & -2\mu \end{bmatrix} \qquad \begin{aligned} P_0 &= P_0 \\ P_1 &= P_0 \frac{2\lambda}{\mu} \\ P_2 &= P_0 \frac{2\lambda^2}{2\mu^2} \end{aligned} \qquad P_m = P_{m-1} \frac{b_{m-1}}{d_m}$$

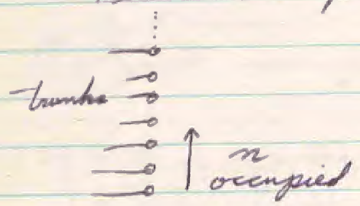
$\boxed{\text{Remember } dt^2 \to 0 \text{ in writing the } A \text{ matrix}}$

$$\sum_{n=0}^{2} P_n = 1 = P_0 \left[1 + 2\frac{\lambda}{\mu} + \left(\frac{\lambda}{\mu}\right)^2\right] = P_0 \frac{1}{\mu^2}\left[\mu^2 + 2\lambda\mu + \lambda^2\right] = \frac{P_0}{\mu^2}(\mu+\lambda)^2$$

$$\boxed{P_0 = \frac{\mu^2}{(\mu+\lambda)^2} \; ; \; P_1 = \frac{2\lambda\mu}{(\mu+\lambda)^2} \; ; \; P_2 = \frac{\lambda^2}{(\mu+\lambda)^2}}$$

Example: Telephone trunks

Initiation of calls Poisson distributed (λ)
Termination of calls exponentially distributed (μ)



$$\boxed{\begin{aligned} b_n &= \lambda \\ d_n &= n\mu \end{aligned}}$$

$$\boxed{P_n = P_0 \prod_{i=1}^{n} \frac{b_{i-1}}{d_i} = P_0 \frac{\lambda^n}{n!\,\mu^n}}$$

$$A = \begin{bmatrix} \cdot & \lambda & & & 0 \to \infty \\ \mu & \cdot & \lambda & & \\ & 2\mu & \cdot & \lambda & \\ & & 3\mu & \cdot & \lambda \\ 0 & & & \ddots & \ddots \end{bmatrix} \downarrow \infty \qquad \sum_{n=0}^{\infty} P_n = 1 = P_0 \sum_{n=0}^{\infty} \frac{\rho^n}{n!} = P_0 e^{\rho} \Rightarrow \boxed{P_0 = e^{-\rho} = e^{-\frac{\lambda}{\mu}}}$$

$$\boxed{P_n = \rho^n \frac{e^{-\rho}}{n!} = \left(\frac{\lambda}{\mu}\right)^n \frac{e^{-\frac{\lambda}{\mu}}}{n!}} \quad \text{Poisson}$$

Limited queues – allow only a finite length:

Maximum number allowed in system is M; what is $P_n(t)$?
Poisson arrivals, exponential service time.

(1) $P_0(t+dt) = P_0(t)\left[1-\lambda dt\right] + P_1(t)\,\mu\, dt$

$$\boxed{\frac{d}{dt} P_0(t) = -\lambda P_0(t) + \mu P_1(t)}$$

(2) $P_n(t+dt) = P_{n-1}(t)\,\lambda\, dt + P_n(t)\left[1-(\lambda+\mu)dt\right] + P_{n+1}(t)\,\mu\, dt$

$$\boxed{\frac{d}{dt} P_n(t) = \lambda P_{n-1}(t) - (\lambda+\mu) P_n(t) + \mu P_{n+1}(t) \quad , \; 0 < n < M}$$

(3) $P_M(t+dt) = P_{M-1}(t)\,\lambda\, dt + P_M(t)\left[1-\mu\, dt\right]$

$$\boxed{\frac{d}{dt} P_M(t) = \lambda P_{M-1}(t) - \mu P_M(t)}$$

Now, we consider only the steady state, so $\frac{d}{dt} = 0$:

$$\boxed{\begin{array}{ll} -\lambda P_0 + \mu P_1 = 0 & n = 0 \\ \lambda P_{n-1} - (\lambda+\mu) P_n + \mu P_{n+1} = 0 & 0 < n < M \\ \lambda P_{M-1} - \mu P_M = 0 & n = M \end{array}}$$

$$A = \begin{bmatrix} -\lambda & \lambda & 0 & 0 & & & \\ \mu & -(\lambda+\mu) & \lambda & & & 0 & \\ 0 & \mu & -(\lambda+\mu) & \lambda & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & & & \lambda & 0 \\ & 0 & & & \mu & -(\lambda+\mu) & \lambda \\ & & & & 0 & \mu & -\mu \end{bmatrix} \qquad \underline{M \times M}$$

For $n \le M$, we can write $P_n = P_0 \left(\frac{\lambda}{\mu}\right)^n = P_0 \rho^n$

$$\sum_{n=0}^{M} P_n = 1 = P_0 \sum_{n=0}^{M} \rho^n = P_0 \left[\sum_{n=0}^{\infty} \rho^n - \sum_{n=M+1}^{\infty} \rho^n \right]$$

$$1 = P_0 \left[\frac{1}{1-\rho} - \rho^{M+1} \frac{1}{1-\rho} \right] = P_0 \frac{1-\rho^{M+1}}{1-\rho}$$

so $P_0 = \dfrac{1-\rho}{1-\rho^{M+1}}$

$$\boxed{P_n = \frac{(1-\rho)\rho^n}{1-\rho^{M+1}} \qquad 0 \le n \le M}$$

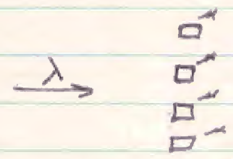
Not that $\rho < 1$, so $\lim_{M \to \infty} P_n = (1-\rho)\rho^n$ ✓

<u>Example</u>: Let $M = 1$

$$P_0 = \frac{1-\rho}{1-\rho^2} = \frac{1}{1+\rho} = \frac{\mu}{\lambda+\mu}$$

$$P_1 = \frac{(1-\rho)\rho}{1-\rho^2} = \frac{\rho}{1+\rho} = \frac{\lambda}{\lambda+\mu}$$

<u>Multi-channel system queue</u>:



One queue, ~~may~~ go to any open channel

n = # in system
λ = arrival rate
μ = service rate for each channel
K = # channels

$$b_n = \lambda, \quad n \ge 0$$

$$d_n = \begin{cases} n\mu, & n \le K \\ K\mu, & n > K \end{cases}$$

$$A = \begin{bmatrix} - & \lambda & & & 0 \\ \mu & - & \lambda & & \\ & 2\mu & & \lambda & \\ & & 3\mu & & \lambda \\ 0 & & & 4\mu & \ddots \\ & & & K\mu & - \quad \lambda \\ & & & & K\mu \quad - \end{bmatrix}$$

$$P_n = P_0 \frac{\rho^n}{n!}, \quad n \le K$$
$$= P_0 \frac{\rho^K}{K!} \left(\frac{\rho}{K}\right)^{n-K} = P_0 \frac{\rho^n}{K!\,K^{n-K}} \qquad n > K$$

Not $\frac{\rho}{K} = \frac{\lambda}{K\mu} < 1$ since $\rho < 1$, $K > 1$

$$1 = P_0 \left[\frac{1-\rho^{K+1}}{1-\rho} + \frac{\rho^{K+1}}{K \cdot K!} \frac{1}{1-K\rho} \right] = ?$$

Distribution of Delay

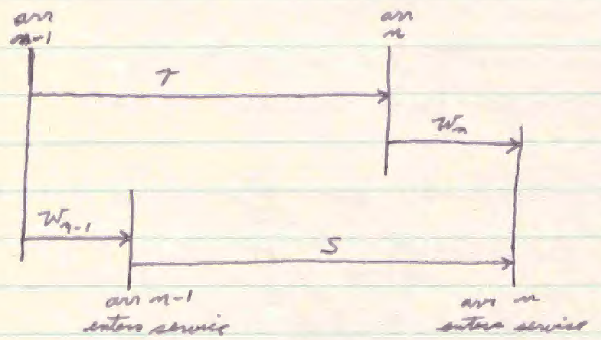
Consider a single channel ~~system~~, first come - first served, infinite queue possible system with arbitrary arrival & service distributions:

Distribution of arrivals is $a(\tau)$; τ = interval between arrivals
Distribution of service or holding time is $h(s)$; s = length of service time.

[The notation $a(\cdot)$ or $h(\cdot)$ is often used when we want to talk about the function itself and not the number $a(\tau)$.]

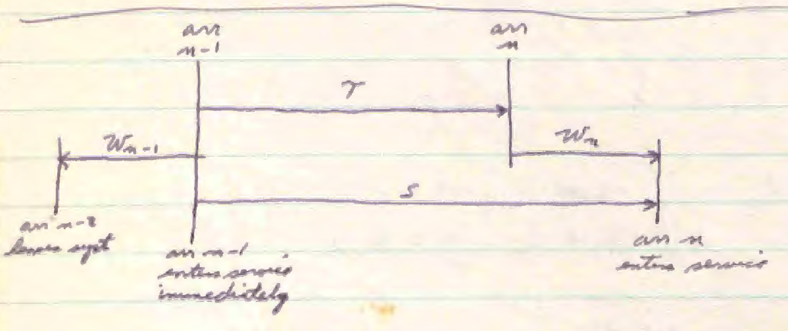
The following is the approach followed by Lindley:

w_n = waiting time of n^{th} arrival = $w_n(t)$.
w_n can be negative, corresponding to the time the system was empty before the n^{th} arrival.



Assuming $w_{n-1} > 0$,

$$\underline{w_n = w_{n-1} + s - \tau}$$



Assuming $w_{n-1} < 0$

$$\underline{w_n = s - \tau}$$

Note that if $w_n < 0$, $s < \tau$ & the system was free for $|s-\tau|$ before n came along. ($w_n = s - \tau < 0$)

Define $\sigma = s - \tau$ = excess of servicetime for $n-1$ over arrival time between $n-1$ and n.

σ is distributed with $r(\sigma)$ & $\sigma = s - \tau$

If $w_{n-1} > 0$, then $w_n = w_{n-1} + \sigma$
$w_{n-1} < 0$, then $w_n = \sigma$

$$r(\sigma) = \int_0^\infty a(x)\, h(x+\sigma)\, dx$$

$$w_n(t) = r(t) \int_{-\infty}^0 w_{n-1}(x)\, dx + \int_0^\infty w_{n-1}(x)\, r(t-x)\, dx$$

$r(t) \int_{-\infty}^0 w_{n-1}(x) dx = P_r\{\text{takes } t \text{ to get served}\} \times P_r\{\text{no wait for arrival } n-1\}$

$\int_0^\infty w_{n-1}(x)\, r(t-x)\, dx = P_r\{$arrival $n-1$ waits x to get served <u>and then</u> takes $t-x$ to get serviced so that $x + (t-x) = t\}$

For a stable system, $w_n(t)$ must approach a limit as $n \to \infty$

$$\lim_{n\to\infty} w_n(t) = w(t)$$

$$w(t) = \int_0^\infty w(x)\, r(t-x)\, dx + r(t) \int_{-\infty}^0 w(x)\, dx$$ if a steady state exists.

Now, define the ~~transform~~ function

$$W(t) = \int_{-\infty}^t w(\xi)\, d\xi = \text{cumulative waiting time}$$

$$\int_{-\infty}^t w(t)\, dt = \int_{-\infty}^t dt \int_0^\infty w(x)\, r(t-x)\, dx + \int_{-\infty}^t dt\, r(t) \int_{-\infty}^0 w(x)\, dx$$

Define $R(t) \equiv \int_{-\infty}^t r(\sigma)\, d\sigma$

$$W(t) = \int_0^\infty dx\, w(x) \int_{-\infty}^{t} dt\, r(t-x) \quad + \quad W(0) \int_{-\infty}^{t} r(t)\,dt$$

$$= \int_0^\infty \underbrace{dx\, w(x)}_{dv}\, \underbrace{R(t-x)}_{u} + \quad W(0)\, R(t)$$

$$= W(x)\, R(t-x) \Big|_{x=0}^{x=\infty} - \int_0^\infty W(x)\, \overbrace{r(t-x)\,dx}^{du} \quad + W(0)\, R(t)$$

$$= -\cancel{W(0) R(t)} - \int_0^\infty W(x)\, r(t-x)\,dx \quad + \cancel{W(0) R(t)} \quad \text{since } R(-\infty) = 0$$

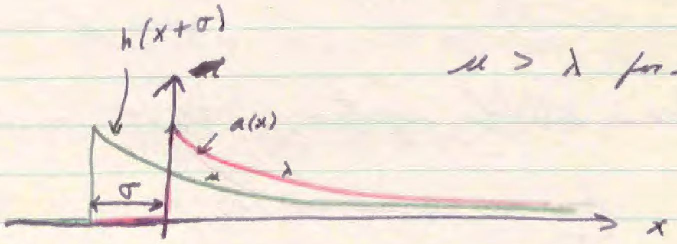
so $$\boxed{W(t) = \int_0^\infty W(x)\, r(t-x)\,dx}$$ <u>Lindley's integral equation for delay time</u>

$\boxed{\text{Example}}$ We cannot solve the above equation in general

Let $a(\tau) = \lambda e^{-\lambda\tau}$

$h(s) = \mu e^{-\mu s}$

$\mu > \lambda$ for stability



$$r(\sigma) = \int_0^\infty a(x)\, h(x+\sigma)\,dx$$

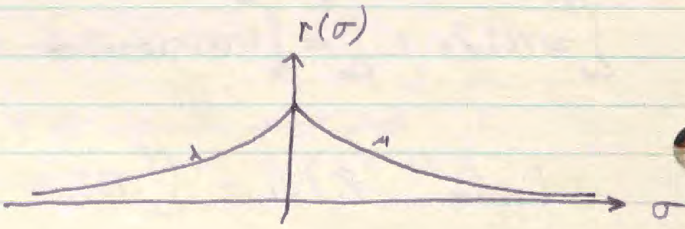
$\sigma \geq 0$, $r(\sigma) = \int_0^\infty \lambda e^{-\lambda x} \mu e^{-\mu(x+\sigma)} dx = \lambda\mu e^{-\mu\sigma} \int_0^\infty dx\, e^{-(\lambda+\mu)\cdot x}$

$\rightarrow r(\sigma) = \dfrac{\lambda\mu}{\lambda+\mu} e^{-\mu\sigma}$

$\sigma < 0$, $r(\sigma) = \int_{-\sigma}^\infty \lambda e^{-\lambda x} \mu e^{-\mu(x+\sigma)} dx = \dfrac{\lambda\mu}{\lambda+\mu} e^{-\mu\sigma} e^{(\lambda+\mu)\sigma}$

$\rightarrow r(\sigma) = \dfrac{\lambda\mu}{\lambda+\mu} e^{\lambda\sigma}$

$$r(\sigma) = \begin{cases} \dfrac{\lambda\mu}{\lambda+\mu} e^{-\mu\sigma}, & \sigma \geq 0 \\ \dfrac{\lambda\mu}{\lambda+\mu} e^{\lambda\sigma}, & \sigma < 0 \end{cases}$$



$$W(t) = \int_0^\infty W(x)\, r(t-x)\, dx$$

$$= \int_0^t W(x) \frac{\lambda\mu}{\lambda+\mu} e^{-\mu(t-x)} dx + \int_t^\infty W(x) \frac{\lambda\mu}{\lambda+\mu} e^{\lambda(t-x)} dx$$

or $$\frac{\lambda+\mu}{\lambda\mu} W(t) = e^{-\mu t}\int_0^t W(x) e^{\mu x} dx + e^{\lambda t}\int_t^\infty W(x) e^{-\lambda x} dx$$

Now, just as we assume a solution (exponential) to solve a differential equation involving exponential terms, we assume a solution for $W(t)$ $[t>0]$ such that $W(0) = 1 - c_1 \neq 0$ and $W(\infty) = 1$:

$$\boxed{\begin{array}{c} W(t) = 1 - c_1 e^{c_2 t} \\ c_2 < 0 \end{array}}$$

Plugging this into the above integral equation, we get

$$\frac{\lambda+\mu}{\lambda\mu}\left[1 - c_1 e^{c_2 t}\right] = e^{-\mu t}\int_0^t \left[1 - c_1 e^{c_2 x}\right] e^{-\mu x} dx + e^{\lambda t}\int_t^\infty \left[1 - c_1 e^{c_2 x}\right] e^{-\lambda x} dx$$

$\boxed{c_2 < 0}$

$$= \frac{1}{\mu} - \frac{c_1}{c_2+\mu} e^{c_2 t} - e^{-\mu t}\left[\frac{1}{\mu} - \frac{c_1}{c_2+\mu}\right] + \frac{1}{\lambda} + \frac{c_1}{c_2-\lambda} e^{c_2 t}$$

Equating the coefficients of $c_1 e^{c_2 t}$ gives

$$-\frac{\lambda+\mu}{\lambda\mu} = \frac{-1}{c_2+\mu} + \frac{1}{c_2-\lambda} = \frac{+(\lambda+\mu)}{(c_2+\mu)(c_2-\lambda)}$$

$$(c_2+\mu)(c_2-\lambda) = -\lambda\mu = c_2^2 + \mu c_2 - \lambda c_2 - \lambda\mu$$

$$c_2^2 + \mu c_2 - \lambda c_2 = 0 \implies \boxed{c_2 = \lambda - \mu}$$

Equating the coefficients of $e^{-\mu t}$ gives

$$0 = \frac{1}{\mu} - \frac{c_1}{c_2+\mu} = \frac{1}{\mu} - \frac{c_1}{\lambda}$$

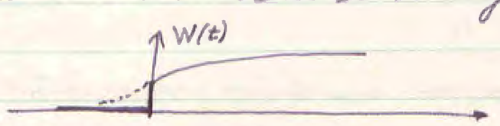
$$\boxed{c_1 = \frac{\lambda}{\mu}}$$

Hence, $W(t) = 1 - C_1 e^{C_2 t}$

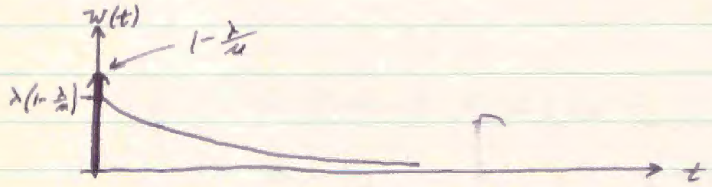
$$\boxed{W(t) = 1 - \frac{\lambda}{\mu} e^{(\lambda-\mu)t} \quad t>0}$$, $\lambda - \mu < 0$ for stability ($\mu > \lambda$)

$W(t) = \int_{-\infty}^{t} w(t)\,dt$ where $w(t)$ = real waiting time

$$\boxed{W(0) = 1 - \frac{\lambda}{\mu} \neq 0}$$



$$w(t) = \left(1 - \frac{\lambda}{\mu}\right)\delta(t) + \frac{\lambda}{\mu}(\mu - \lambda)\, e^{(\lambda-\mu)t}$$



This is only real waiting time & does not consider the free time of the system.

State Probability Approach :

$w(t)\,dt$ = Pr {there is a positive waiting time between t & $dt + t$} = (but first of t) x dt
t = w.t.

$= \sum_{n=1}^{\infty} \Pr\{A(n) \;\&\; B(n) \;\&\; C\}$

$A(n)$: n units in system ($n \geq 1$)
$B(n)$: $n-1$ services in time t
C : service in time dt

$$\left.\begin{aligned} P(ABC) &= P(AB)P(C) \\ P(AB) &= P(A)P(A|B) \end{aligned}\right\} \quad P(ABC) = P(A)\,P(B|A)\,P(C)$$

$P(A) = \left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^{n}$ see p. 65 , $\left(\frac{\lambda}{\mu}\right) \equiv \rho$

$P(B|A) = \frac{(\mu t)^{n-1}}{(n-1)!} e^{-\mu t}$ see p. 65, p. 69 , assumed exp serv times.

$P(C) = \mu\,dt$

$$w(t) = \mu\left(1-\frac{\lambda}{\mu}\right)\frac{\lambda}{\mu} e^{-\mu t} \sum_{n=1}^{\infty} \frac{(\lambda t)^{n-1}}{(n-1)!} = (\mu-\lambda)\frac{\lambda}{\mu} e^{-\mu t} \sum_{k=0}^{\infty} \frac{(\lambda t)^{k}}{k!}$$

$$= (\mu-\lambda)\frac{\lambda}{\mu} e^{-\mu t} e^{+\lambda t} = \underline{\underline{(\mu-\lambda)\frac{\lambda}{\mu} e^{(\lambda-\mu)t}}}$$

Same as obtained before;

$$Pr\{t>0\} = \int_0^\infty w(t)\,dt = \frac{\lambda}{\mu}$$

$$\boxed{w(t|t>0) = \frac{w(t)}{Pr\{t>0\}} = \frac{(\mu-\lambda)\frac{\lambda}{\mu} e^{(\lambda-\mu)t}}{\lambda/\mu} = (\mu-\lambda)e^{(\lambda-\mu)t}}$$

$$\boxed{E(t|t>0) = \frac{1}{\mu-\lambda} \quad \Rightarrow \quad E(t) = \frac{\lambda}{\mu}\,\frac{1}{\mu-\lambda}}$$

Distribution of total time spent in system:

y = total time spent in the system by an arrival, distributed by $v(y)$

$v(y)\,dy = Pr\{\text{no wait}\}\, Pr\{\text{service time between } y \;\&\; dy+y\}$

$\qquad + \int dt\, Pr\{\text{wait for } t \;\&\; \text{then have service time between } y-t \;\&\; y-t+dy\}$

$$v(y) = \int_0^\infty w(t)\, h(y-t)\,dt$$

Laplace transforms $V(s)$, $W(s)$, $H(s)$ are such that

$$V(s) = W(s)\,H(s)$$

$$W(s) = \left(1-\frac{\lambda}{\mu}\right) + \frac{\frac{\lambda}{\mu}(\mu-\lambda)}{s+\mu-\lambda} \quad ; \quad H(s) = \frac{\mu}{s+\mu}$$

$$V(s) = \frac{\mu-\lambda}{s+\mu} + \frac{\lambda(\mu-\lambda)}{s+\mu-\lambda} = \frac{\mu-\lambda}{s+\mu-\lambda}$$

$\boxed{v(y) = (\mu-\lambda)e^{(\lambda-\mu)y}}$ Total time vs. waiting time $w(t)$

$$\boxed{E(t|t>0) = E(y) = \frac{1}{\mu-\lambda}}$$

Continuous Markov Processes:

Reviewing the fundamentals of discrete processes, we recall that

$$\pi_j(n+1) = \sum_{i=1}^{N} \pi_i(n) P_{ij}$$

so $\pi_j(n+1) - \pi_j(n) = \sum_{i=1}^{N} \pi_i(n)\left[P_{ij} - \delta_{ij}\right] = \sum_{i=1}^{N} \pi_i(n) a_{ij}$

where $\boxed{a_{ij} \equiv P_{ij} - \delta_{ij}}$

$$\sum_{j=1}^{N} P_{ij} = 1 \Rightarrow \sum_{j=1}^{N} a_{ij} = 0$$

Hence, the matrix $\boxed{(a_{ij}) = A \text{ is a differential matrix}}$, and we can write

$$\boxed{\Delta \underline{\pi}(n) = \pi(n) A}$$

Now, following an analogous development for continuous processes, we say

$$\pi_j(t+dt) = \sum_{i=1}^{N} \pi_i(t) P_{ij}(dt)$$

Now, define a_{ij} such that $\boxed{P_{ij}(dt) \equiv a_{ij}\, dt}$, $i \neq j$

<u>a_{ij} = rate of transition from state i to state j.</u>

Since we cannot speak of transitions from i to i, we define

$\rightarrow$ <u>$a_{ii} = -\sum_{j\neq i} a_{ij}$ so that $P_{ii}(dt) = 1 - \sum_{j\neq i} a_{ij}\, dt = 1 + a_{ii}\, dt$</u>

Then we can write in general that

$$\boxed{P_{ij}(dt) = \delta_{ij} + a_{ij}\, dt \quad ; \text{all } i,j}$$

so

$$\pi_j(t+dt) = \sum_{i=1}^{N} \pi_i(t)\left[\delta_{ij} + a_{ij}\, dt\right] = \pi_j(t) + \sum_{i=1}^{N} \pi_i(t) a_{ij}\, dt$$

$$\lim_{dt\to 0} \frac{\Pi_j(t+dt)-\Pi_j(t)}{dt} = \boxed{\frac{d}{dt}\pi_j(t) = \sum_{i=1}^{N}\Pi_i(t)\, a_{ij}}$$

Writing this result in matrix form gives

$$\boxed{\frac{d}{dt}\,\underline{\Pi}(t) = \underline{\Pi}(t)\, A}$$ cf p 89.

<u>Note</u>: The problem is completely specified by the off-diagonal elements of A. We have chosen the diagonal elements for ease of notation.

<u>State probabilities conditioned on previous state of system</u>:

$\boxed{\text{Discrete}}$: $P_{ij}(m,n) \equiv Pr\{\text{in state } j \text{ at time } n \mid \text{in state } i \text{ at time } m\}$

$$\Pi_j(n) = \sum_{i=1}^{N}\Pi_i(m)\, P_{ij}(m,n) \qquad \underline{n > m}$$

$$\underline{\Pi}(n) = \underline{\Pi}(m)\; P(m,n)$$

For stationary states (no transients), this reduces to

$$\boxed{\underline{\Pi}(n) = \underline{\Pi}(m)\; P(n-m)}$$

And if P is independent of time, $\longleftrightarrow$ $\boxed{\underline{\Pi}(n) = \underline{\Pi}(m)\; P^{n-m}}$ c.f. p. 42

$\boxed{\text{Continuous}}$: $P_{ij}(\tau,t)$ = distribution function of $Pr.\{\text{in } S_j \text{ at } t \mid \text{in } S_i \text{ at } \tau\}$

$$\pi_j(t) = \sum_{i=1}^{N}\pi_i(\tau)\; P_{ij}(\tau,t)$$

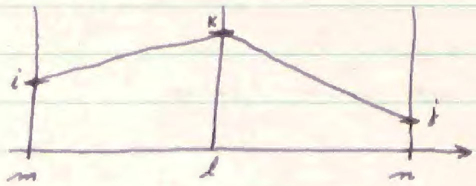
$$\boxed{\underline{\Pi}(t) = \underline{\Pi}(\tau)\; P(\tau,t)}$$

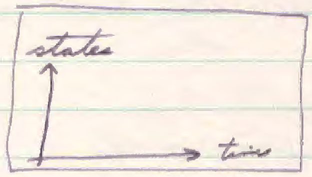
For stationary states, this becomes

$$\boxed{\underline{\Pi}(t) = \underline{\Pi}(\tau)\; P(t-\tau)}$$

Chapman - Kolmogorov Relations:

Discrete:





We can write $P_{ij}(m,n)$ as a combination of functions $P_{ik}(m,l)$ and $P_{kj}(l,n)$ by the relation

$$\boxed{P_{ij}(m,n) \equiv \sum_{k=1}^{N} P_{ik}(m,l)\, P_{kj}(l,n)}$$

where the indicated summation is over all values of k and the transition probabilities P_{ij} are independent for all i,j.

Now, let us consider the back end of the ~~variable~~ interval; i.e., $l = n-1$

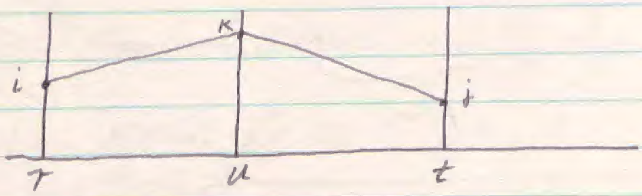
$$P_{ij}(m,n) = \sum_{k=1}^{N} P_{ik}(m,n-1)\, P_{kj}(n-1,n) = \sum_{k=1}^{N} P_{ik}(m,n-1)\, P_{kj}(n-1)$$

where $P_{kj}(n-1)$ denotes the transition probability from k to j at time $n-1$; this is not the $P(n-m)$ considered on the previous page. In matrix notation:

$$P(m,n) = P(m,n-1)\, P(n-1) \quad + \quad P(m,n+1) = P(m,n)\, P(n)$$

$$\boxed{\begin{aligned} &\Delta_n P(m,n) = P(m,n)\, A(n) = P(m,n+1) - P(m,n) \\ &\text{where } A(n) = P(n) - I \end{aligned}}$$

Continuous processes:



$$P_{ij}(T,t) = \sum_{k=1}^{N} P_{ik}(T,u)\, P_{kj}(u,t)$$

Let $\underline{u = t - dt}$; then, $P_{ij}(T,t) = \sum_{k=1}^{N} P_{ik}(T, t-dt)\, \underbrace{P_{kj}(t-dt, t)}_{P_{kj}(dt)}$

$$P_{ij}(T,t) = \sum_{k=1}^{N} P_{ik}(T, t-dt)\left[\delta_{kj} + a_{kj}(t-dt)\,dt\right]$$

$$P_{ij}(T,t) - P_{ij}(T,t-dt) = \sum_{k=1}^{N} P_{ik}(T,t-dt)\, a_{kj}(t-dt)\,dt$$

$$\boxed{\begin{aligned} &\frac{\partial}{\partial t} P_{ij}(T,t) = \sum_{k=1}^{N} P_{ik}(T,t)\, a_{kj}(t) \\ \text{or}\quad &\frac{\partial}{\partial t} P(T,t) = P(T,t)\, A(t) \end{aligned}}$$

Note that if $A(t) = A$, and $T = 0$, the above equations reduce to:

$\frac{d}{dt} \underline{\Pi}(t) = \underline{\Pi}\, \cancel{(t)}\, A$ cf p. 87

The equations on these two pages are called the "forward" equations as they are written at the forward end of the interval.

We can also write equations at the "backward" end of the interval, yielding the "backward" equations:

Backward equations:

Discrete: $l = m+1 \Rightarrow P_{ij}(m,n) = \sum_{k=1}^{N} \underbrace{P_{ik}(m,m+1)}_{P(m)} P_{kj}(m+1,n)$

$$P(m,n) = P(m)\, P(m+1,n)$$

$$\boxed{\begin{aligned} \Delta_m P(m,n) &= P(m+1,n) - P(m,n) = -A(m)\,P(m+1,n) \\ A(m) &= P(m) - I \end{aligned}}$$

Continuous: $u = \tau + d\tau \Rightarrow P_{ij}(\tau,t) = \sum_{k=1}^{N} P_{ik}(\tau,\tau+d\tau)\, P_{kj}(\tau+d\tau,t)$

$$P_{ij}(\tau,t) = \sum_{k=1}^{N}\left[\delta_{ik} + a_{ik}(\tau)\,d\tau\right] P_{kj}(\tau+d\tau,t)$$

$$\frac{\partial}{\partial \tau} P_{ij}(\tau,t) = -\sum_{k=1}^{N} a_{ik}(\tau)\, P_{kj}(\tau,t)$$

$$\boxed{\frac{\partial}{\partial \tau} P(\tau,t) = -A(\tau)\,P(\tau,t)}$$

Note: $P_{ij}(\tau,t) \geq 0$, but $\sum_{j=1}^{N} P_{ij}(\tau,t) \leq 1$

The reason is that the system may be able to kill itself off before time t. These cases are, however, rare.

Common solution to forward & backward equations:

Discrete:

$$\left.\begin{aligned} \Delta_m P(m,n) &= P(m,n)\,A(n) \\ P(m,n) &= P(m,n-1)\,P(n-1) \end{aligned}\right\} \quad \boxed{P(m,n) = \prod_{k=m}^{n-1} P(k) = P(m)\,P(m+1)\ldots P(n)\,P(n-1)}$$

As a check for the solution, plug into $P(m,n) = P(m,n-1)\,P(n-1)$

$$\prod_{k=m}^{n-1} P(k) \overset{?}{=} \prod_{k=m}^{n-2} P(k)\; P(n-1) \quad \checkmark$$

Relation of solution to $\underline{\Pi}(m)$

$$\underline{\Pi}(m)\, P(m,n) = \underline{\Pi}(n)$$

$$\underline{\Pi}(n) = \underline{\Pi}(m) \prod_{k=m}^{n-1} P(k)$$

If $m = 0$, $\quad \underline{\Pi}(n) = \underline{\Pi}(0) \prod_{k=0}^{n-1} P(k)$

If $P(k) = P$, $\quad \underline{\Pi}(n) = \underline{\Pi}(0)\, P^{n}$ ✓

Check on backward equations:

(1) $\Delta_m P(m,n) = -A(m)\, P(m+1,n)$

(2) $P(m,n) = P(m)\, P(m+1,n)$

Plugging the above solution for $P(m,n)$ into (2) gives:

$$\prod_{k=m}^{n-1} P(k) = P(m) \prod_{k=m+1}^{n-1} P(k) = \prod_{k=m}^{n-1} P(k)$$ ✓

Continuous variables:

$$\frac{\partial}{\partial t} P(\tau,t) = \lim_{dt\to 0} \frac{P(\tau,t+dt) - P(\tau,t)}{dt} = P(\tau,t)\, A(t)$$

Assume solution $P(\tau,t) = e^{\int_\tau^t A(x)dx}$ = matrix function

$$\frac{\partial}{\partial t} P(\tau,t) = \lim_{dt\to 0} \frac{1}{dt}\left[e^{\int_\tau^{t+dt} A(x)dx} - e^{\int_\tau^{t} A(x)dx} \right]$$

$$= \lim_{dt\to 0} \frac{1}{dt}\left[e^{\int_\tau^{t} A(x)dx}\, e^{\int_t^{t+dt} A(x)dx} - e^{\int_\tau^{t} A(x)dx} \right]$$

$$= \lim_{dt\to 0} \frac{1}{dt}\, e^{\int_\tau^{t} A(x)dx} \left[e^{\int_t^{t+dt} A(x)dx} - I \right] =$$

$$\frac{\partial}{\partial t} P(\tau,t) = e^{\int_\tau^t A(x)dx} \lim_{dt\to 0} \frac{1}{dt}\left[I + \int_t^{t+dt} A(x)dx + \cdots - I\right]$$

$$= e^{\int_\tau^t A(x)dx} \lim_{dt\to 0} \frac{1}{dt} A(t)dt = e^{\int_\tau^t A(x)dx} A(t)$$

$$= P(\tau,t)\, A(t)$$

For the backward equations, same solution plugs into

$$\frac{\partial}{\partial \tau} P(\tau,t) = -A(\tau)\, P(\tau,t)$$

Assuming the same solution, $P(\tau,t) = e^{\int_\tau^t A(x)dx}$, we have

$$\frac{\partial}{\partial \tau} P(\tau,t) = \frac{\partial}{\partial \tau} e^{\int_\tau^t A(x)dx} = \lim_{d\tau\to 0} \frac{1}{d\tau}\left[e^{\int_{\tau+d\tau}^t A(x)dx} - e^{\int_\tau^t A(x)dx}\right]$$

$$= \lim_{d\tau\to 0} \frac{1}{d\tau}\left[+e^{\int_{\tau+d\tau}^{t} A(x)dx} - e^{\int_\tau^t A(x)dx}\right] = \cancel{\lim_{d\tau\to 0} \frac{1}{d\tau}\left[e^{\int_t^{\tau_0} A(x)dx}\, e\right]}$$

$$= \lim_{d\tau\to 0} \frac{1}{d\tau}\left[+e^{\int_{\tau+d\tau}^{\tau} A(x)dx}\, e^{\int_\tau^t A(x)dx} - e^{\int_\tau^t A(x)dx}\right] = \lim_{d\tau\to 0} \frac{1}{d\tau}\left[e^{\int_{\tau+d\tau}^{\tau} A(x)dx} - I\right] e^{\int_\tau^t A(x)dx}$$

$$= \lim_{d\tau\to 0} \frac{1}{d\tau}\left[I + \int_{\tau+d\tau}^{\tau} A(x)dx + \cdots - I\right] e^{\int_\tau^t A(x)dx}$$

$$= \lim_{d\tau\to 0} \frac{1}{d\tau}\left[-A(\tau)d\tau\right] e^{\int_\tau^t A(x)dx} = -A(\tau)\, e^{\int_\tau^t A(x)dx} = -A(\tau)\, P(\tau,t)$$