6.536

SYSTEMS ENGINEERING
AND
OPERATIONS RESEARCH

VOL. II

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N 2739

Examples:

Let the transition probability matrix be $P(n) = \begin{Bmatrix} P_e, & n \text{ even} \\ P_o, & n \text{ odd} \end{Bmatrix}$

where P_e & P_o are constant matrices

$$P_e = \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix} \quad , \quad P_o = \begin{bmatrix} 1-p' & p' \\ q' & 1-q' \end{bmatrix}$$

Now, $\underline{\pi}(n+1) = \underline{\pi}(n)\,P(n)$

$$\underline{\pi}(1) = \underline{\pi}(0)\,P_e$$
$$\underline{\pi}(2) = \underline{\pi}(1)\,P_o = \underline{\pi}(0)\,P_e P_o$$
$$\underline{\pi}(3) = \underline{\pi}(2)\,P_e = \underline{\pi}(0)\,P_e P_o P_e$$

Now, if we define $P_{eo} \equiv P_e P_o$, we have

$$\underline{\pi}(n) = \begin{cases} \underline{\pi}(0)\,(P_e P_o)^{n/2} = \underline{\pi}(0)\,P_{eo}^{\,n/2}, & n \text{ even} \\ \underline{\pi}(0)\,P_{eo}^{\frac{n-1}{2}}\,P_e, & n \text{ odd} \end{cases}$$

Assume a steady state solution exists $\underline{\pi} = \begin{Bmatrix} \underline{\pi}_e, & n \text{ even} \\ \underline{\pi}_o, & n \text{ odd} \end{Bmatrix}$

$$\boxed{\underline{\pi}_e = \underline{\pi}_e\,P_{eo}}$$

$$\underline{\pi}_e\,P_e = \underline{\pi}_e\,P_{eo}P_e = \underline{\pi}_o$$

but $\underline{\pi}_e\,P_{eo}P_e = (\underline{\pi}_e\,P_e)P_oP_e = \underline{\pi}_o\,P_{oe}$ where $P_{oe} \equiv P_o P_e$

so $\boxed{\underline{\pi}_o = \underline{\pi}_o\,P_{oe}}$

Let $P_e = \begin{bmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{bmatrix}$, $P_o = \begin{bmatrix} 2/3 & 1/3 \\ 3/4 & 1/4 \end{bmatrix}$

$P_{eo} = P_e P_o = \begin{bmatrix} \frac{11}{16} & \frac{5}{16} \\ \frac{17}{24} & \frac{7}{24} \end{bmatrix}$, $P_{oe} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{11}{16} & \frac{5}{16} \end{bmatrix}$

For a two state system, with transition matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the steady state probabilities can be found to be

$\pi_1 = \frac{c}{c+b}$ and $\pi_2 = \frac{b}{c+b}$

In this case, $\pi_1 = \frac{34}{49}$, $\pi_2 = \frac{15}{49}$, for P_{eo}

$\pi_1 = \frac{33}{49}$, $\pi_2 = \frac{16}{49}$, for P_{oe}

So $\pi_e = \begin{bmatrix} 34/49 & \frac{15}{49} \end{bmatrix}$

$\pi_o = \begin{bmatrix} 33/49 & 16/49 \end{bmatrix}$

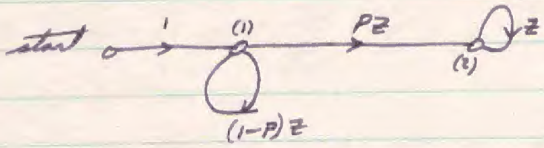
As a check on these answers, $\left\{ \begin{array}{l} \pi_e = \pi_o P_o \\ \pi_o = \pi_e P_e \end{array} \right\}$

	P_e alone	P_o alone	n even	n odd
π_1	0.667	0.692	0.694	0.674
π_2	0.333	0.308	0.306	0.326

There is no simple relation between the results if we used one constant matrix ~~with or~~ and the results if we used the same matrix alternated with another.

Hunting problem:

Probability p that we will find what we are looking for at any discrete time n:



$$P = \begin{bmatrix} 1-p & p \\ 0 & 1 \end{bmatrix}$$

The transmission through node (1) is the probability that we have not found it at time n:

$$\Pi_1(z) = \frac{1}{1-(1-p)z}$$

$$\boxed{\pi_1(n) = (1-p)^n \ , \ n \geq 0} \quad \text{so} \quad \boxed{\pi_2(n) = 1-(1-p)^n}$$

$$\lim_{n\to\infty} \pi_1(n) = 0 \ , \ p \neq 0 \Rightarrow \text{we will eventually find it.}$$

$$\underline{\pi}(n+1) = \underline{\pi}(n) P \quad ;$$

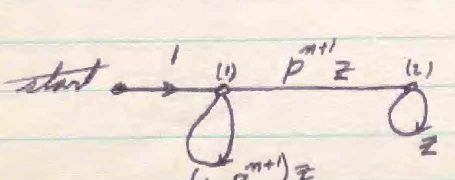
Assume we haven't found it at $n=0$; $\underline{\pi}(0) = [1 \quad 0]$

$$\underline{\pi}(1) = \underline{\pi}(0) P = [1-p \quad p]$$

$$\underline{\pi}(2) = \underline{\pi}(1) P = [(1-p)^2 \quad p(2-p)]$$

$$\lim_{n\to\infty} \underline{\pi}(n) = [0 \quad 1]$$

Now, assume we get tired as we hunt, so we are less likely to find "it":



$$P(n) = \begin{bmatrix} 1-p^{n+1} & p^{n+1} \\ 0 & 1 \end{bmatrix}$$

Non-stationary processes

$\underline{\Pi}(n+1) = \underline{\Pi}(n)\, P(n)$

$\underline{\Pi}(0) = [1 \quad 0]$

$\underline{\Pi}(1) = \underline{\Pi}(0)\, P(0) = [1-p \quad p]$

$\underline{\Pi}(2) = \underline{\Pi}(1)\, P(1) = [(1-p)(1-p^2) \qquad (1-p)p^2 + p]$

$\underline{\Pi}(3) = \underline{\Pi}(2)\, P(2) = [(1-p)(1-p^2)(1-p^3) \qquad (1-p)(1-p^2)p^3 + (1-p)p^2 + p]$

$\boxed{\Pi_1(n) = \prod_{i=1}^{n} (1-p^i)} \quad + \quad \boxed{\Pi_2(n) = 1 - \Pi_1(n)}$

Note that $\Pi_1(n)$ for this stationary (time invariant) system is always less than $\Pi_1(n)$ for the non-stationary because $p<1$, so $(1-p^i) > 1-p$.

Limiting state probabilities:

For the time-invariant or stationary process, $\lim_{n\to\infty} \Pi_1(n) = 0$.

For the non-stationary process, the limit is not obvious

Since $p<1$, $\lim_{i\to\infty} (1-p^i) = 1$

$\Pi_1(\infty) = \prod_{i=1}^{\infty} (1-p^i)$

For easier manipulation, we reduce this to a summation

$\ln \Pi_1(\infty) = \sum_{i=1}^{\infty} \ln(1-p^i)$

Now, $\frac{1}{1-x} = 1 + x + x^2 + \dots$ for $|x| < 1$

$$\int_0^x \frac{1}{1-x}\,dx = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots = -\ln(1-x)\Big|_{x=0}^{x=x}$$

$$-\ln(1-p^i) = p^i + \frac{p^{2i}}{2} + \frac{p^{3i}}{3} + \cdots$$

$$-\ln \pi_1(\infty) = -\sum_{i=1}^{\infty}\left(p^i + \tfrac{1}{2}p^{2i} + \tfrac{1}{3}p^{3i} + \cdots\right)$$

$$= \frac{p}{1-p} + \frac{1}{2}\,\frac{p^2}{1-p^2} + \frac{1}{3}\,\frac{p^3}{1-p^3} + \cdots$$

$$\boxed{-\ln \pi_1(\infty) = \sum_{j=1}^{\infty} \frac{p^j}{j(1-p^j)}}$$

We cannot directly find a closed form for this expression, but we can set bounds:

<u>Upper bound</u>: $-\pi_1(\infty) > \sum_{j=1}^{K} \frac{p^j}{j(1-p^j)}$ for any finite K

In particular, let $K = 2$:

$$-\ln \pi_1(\infty) > \frac{p}{1-p} + \frac{1}{2}\,\frac{p^2}{1-p^2} = \frac{p(1+p) + \frac{1}{2}p^2}{1-p^2}$$

$$-\ln \pi_1(\infty) > \frac{p + \frac{3}{2}p^2}{1-p^2}$$

or $\pi_1(\infty) < \exp\left\{-\frac{p(1-\frac{3}{2}p)}{1-p^2}\right\}$

If $p = 1/2$, $\pi_1(\infty) < e^{-\frac{7}{6}}$

Lower bound:

$$-\ln \pi_1(\infty) = \sum_{j=1}^{\infty} \frac{p^j}{j(1-p^j)}$$

but $1-p^j > 1-p$ for $j > 1$, $p < 1$.

so $$-\ln \pi_1(\infty) < \frac{1}{1-p} \sum_{j=1}^{\infty} \frac{p^j}{j} = \frac{1}{1-p}\left[-\ln(1-p)\right]$$

$$\pi_1(\infty) > (1-p)^{\frac{1}{1-p}} \rightarrow \left(\tfrac{1}{2}\right)^2 = \tfrac{1}{4} \quad \text{for } p = 2$$

Examples

p	lower bound $= (1-p)^{\frac{1}{1-p}}$	upper bound $= \exp\left\{-\frac{p(1+\frac{3}{2}p)}{1-p^2}\right\}$
0	1	1
1/4	0.681	0.694
1/2	0.250	0.312
3/4	0.0039	0.026
1	0	0

Matrix Functions:

For the continuous-time Markov process, we have the relation

$$\dot{\underline{\pi}}(t) = \underline{\pi}(t)\, A(t)$$

with the general solution $\quad \underline{\pi}(t) = \underline{\pi}(0)\, e^{\int_0^t A(t)\,dt}$

If $A(t) = A$ is independent of time, we can define

$$e^{At} = I + At + A^2 \frac{t^2}{2!} + A^3 \frac{t^3}{3!} + \cdots$$

in analogy with scalar functions.

For certain transition probability matrices, this expansion can be put into closed form:

Example: Machine, repairman $\left\{ \begin{array}{l} \text{breakdown rate} = \lambda \\ \text{repair rate} = \mu \end{array} \right\}$

$$A = \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix}$$

$$A^2 = \begin{bmatrix} \lambda^2 + \lambda\mu & -\lambda^2 - \lambda\mu \\ -\lambda\mu - \mu^2 & \lambda\mu + \mu^2 \end{bmatrix} = -(\lambda+\mu)\, A$$

Hence $\quad e^{At} = I + At - \frac{1}{2!}(\lambda+\mu)At^2 + \frac{1}{3!}(\lambda+\mu)^2 At^3 + \cdots$

$$= I + A\,\frac{1}{\lambda+\mu}\left[1 - e^{-(\lambda+\mu)t}\right]$$

Hence $\quad \underline{\pi}(t) = \underline{\pi}(0) + \underline{\pi}(0)\, A\, \dfrac{1 - e^{-(\lambda+\mu)t}}{\lambda+\mu}$

If $\underline{\pi}(0) = [1 \;\; 0]$,

$$\boxed{\begin{aligned} \pi_1(t) &= \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu}\, e^{-(\lambda+\mu)t} \\ \pi_2(t) &= \frac{\lambda}{\lambda+\mu} - \frac{\lambda}{\lambda+\mu}\, e^{-(\lambda+\mu)t} \end{aligned}}$$

c.f. pp. $\left\{\begin{matrix}65\\66\end{matrix}\right\}$

Generalized Matrix Functions:

Consider an $N \times N$ matrix B & we want to find $f(B)$, knowing the function (or operator) $f(\cdot)$.

Define the N forward (row) eigen vectors of B by:

$$\underline{x}^i B = \lambda_i \underline{x}^i$$

and define the N rearward (column) eigen vectors of B by:

$$B \underline{y}^i = \lambda_i \underline{y}^i$$

The λ_i are the N characteristic values of B given by

$$\det (B - \lambda_i I) = 0$$

For a stochastic matrix, all λ_i are distinct.
The eigen vectors have the property that

$$\underline{x}^i \underline{y}^j = \begin{Bmatrix} 0, & i \neq j \\ \alpha_i, & i = j \end{Bmatrix}$$

We can normalize the eigenvectors by setting

$$\boxed{\underline{x}^i \underline{y}^i = 1}$$

Now, define $\boxed{M_i = \underline{y}^i \underline{x}^i}$

The desired function is

$$\boxed{f(B) = \sum_{i=1}^{N} f(\lambda_i) M_i}$$

Example: $P = \begin{bmatrix} 3/4 & 1/4 \\ 1/3 & 2/3 \end{bmatrix}$

$$\begin{vmatrix} \frac{3}{4}-\lambda & \frac{1}{4} \\ \frac{1}{3} & \frac{2}{3}-\lambda \end{vmatrix} = 0 \longrightarrow \lambda = 1, \frac{5}{12} \qquad \boxed{\begin{matrix}\lambda_1 = 1 \\ \lambda_2 = 5/12\end{matrix}}$$

For a stochastic matrix we will always have $\lambda = 1$

$$\underline{x}^1 P = \lambda_1 \underline{x}^1 = \underline{x}^1 = [a \ b]$$

$$[a \ b] \begin{bmatrix} 3/4 & 1/4 \\ 1/3 & 2/3 \end{bmatrix} = [a \ b] \longrightarrow \left.\begin{matrix} a = 4 \\ b = 3 \end{matrix}\right\} \text{within a constant multiplier}$$

so $\underline{x}^1 = [4 \ \ 3]$

$$\underline{x}^2 P = \frac{5}{12} \underline{x}^2 \longrightarrow \underline{x}^2 = [1 \ \ -1]$$

$$P \underline{y}^1 = \lambda_1 \underline{y}^1$$

$$\begin{bmatrix} 3/4 & 1/4 \\ 1/3 & 2/3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \longrightarrow a = b = 1$$

$$\underline{y}^1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$P \underline{y}^2 = \frac{5}{12} \underline{y}^2 \rightarrow \begin{bmatrix} 3/4 & 1/4 \\ 1/3 & 2/3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \frac{5}{12} \begin{bmatrix} a \\ b \end{bmatrix} \rightarrow \begin{matrix} a = 3 \\ b = -4 \end{matrix}$$

$$\underline{y}^2 = [3 \ \ -4]$$

Normalizing, we have

$$\left.\begin{matrix} \underline{x}^1 \underline{y}^1 = 7 \rightarrow \underline{x}^1 = [4/7 \ \ 3/7] , \quad \underline{y}^1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ \underline{x}^2 \underline{y}^2 = 7 \longrightarrow \underline{x}^2 = [1 \ \ -1] , \quad \underline{y}^2 = \begin{bmatrix} 3/7 \\ -4/7 \end{bmatrix} \end{matrix}\right\} \text{We can normalize either } \underline{x}^i \text{ or } \underline{y}^i$$

$$M_1 = \underset{\sim}{y}^1 \underset{\sim}{x}^1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 4/7 & 3/7 \end{bmatrix} = \begin{bmatrix} 4/7 & 3/7 \\ 4/7 & 3/7 \end{bmatrix}$$

$$M_2 = \underset{\sim}{y}^2 \underset{\sim}{x}^2 = \begin{bmatrix} 3/7 \\ -4/7 \end{bmatrix} \begin{bmatrix} 1 & -1 \end{bmatrix} = \begin{bmatrix} 3/7 & -3/7 \\ -4/7 & 4/7 \end{bmatrix}$$

If $\lambda_1 = 1$, then the rows of M_1 are the ~~steady~~ state probabilities of the system. Also, all other M_i $(i \neq 1)$ are differential matrices, while M_1 is stochastic.

Now, $\boxed{f(P) = f(1) M_1 + f(5/12) M_2}$ $\boxed{\text{discrete time}}$

If $A = P - I$, $\begin{cases} \text{the } M_i \text{ are the same as for } P \\ \text{the } \lambda_i \text{ are decreased by 1 from the values for } P. \end{cases}$

so $\boxed{f(A) = f(0) M_1 + f(-\tfrac{7}{12}) M_2}$ $\boxed{\text{continuous time}}$

E.g.: if $\underline{\pi}(n) = \underline{\pi}(0) P^n$

$$P^n = f(P) = (1)^n M_1 + \left(\tfrac{5}{12}\right)^n M_2$$

$$P^n = \begin{bmatrix} 4/7 & 3/7 \\ 4/7 & 3/7 \end{bmatrix} + \left(\tfrac{5}{12}\right)^n \begin{bmatrix} 3/7 & -3/7 \\ -4/7 & 4/7 \end{bmatrix}$$

(Can't handle multi-chain problems.)

M_1 always has identical rows (state probabilities)

For continuous time $\dot{\underline{\pi}}(t) = \underline{\pi}(t) A$, $A = P - I = \begin{bmatrix} -1/4 & 1/4 \\ 1/3 & -1/3 \end{bmatrix}$

$$\underline{\pi}(t) = \underline{\pi}(0) e^{At}$$

$$\underline{\pi}(t) = \underline{\pi}(0) \left[e^0 \begin{bmatrix} 4/7 & 3/7 \\ 4/7 & 3/7 \end{bmatrix} + e^{-7/12\, t} \begin{bmatrix} 3/7 & -3/7 \\ -4/7 & 4/7 \end{bmatrix} \right]$$

Discrete – continuous :

If $\underline{\pi}(0)\, e^{At} = \underline{\pi}(0)\, P^{n}$,

then when $t = n$, $\boxed{P = e^{A}}$

This connects the discrete & continuous cases as in the analogy of the Z- and Laplace ~~transform~~ transforms. Cf. p. 55.

Example: Time-varying Markov process.

$$\underline{\pi}(t) = \underline{\pi}(0)\, e^{\int_0^t A(t)\,dt}$$

Let
$$A(t) = \begin{bmatrix} -\lambda e^{-at} & \lambda e^{-at} \\ \mu e^{-at} & -\mu e^{-at} \end{bmatrix} = e^{-at} \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix} = e^{-at} A.$$

$$\underline{\pi}(t) = \underline{\pi}(0)\, e^{A\int_0^t e^{-at}\,dt} = \underline{\pi}(0)\, e^{\frac{A}{a}(1-e^{-at})}$$

$\lambda_1 = 0 \quad \leftrightarrow$ stochastic $\lambda = 1$
$\lambda_2 = -(\lambda+\mu)$

$\underline{x}^1 = [\mu \quad \lambda] \qquad \underline{y}^1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\underline{x}^2 = [1 \quad -1] \qquad \underline{y}^2 = \begin{bmatrix} \lambda \\ -\mu \end{bmatrix}$

$\underline{X}^1 = \left[\frac{\mu}{\lambda+\mu} \quad \frac{\lambda}{\lambda+\mu}\right] \qquad \underline{y}^1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\underline{x}^2 = [1 \quad -1] \qquad \underline{y}^2 = \begin{bmatrix} \frac{\lambda}{\lambda+\mu} \\ -\frac{\mu}{\lambda+\mu} \end{bmatrix}$

$$M_1 = \begin{bmatrix} \frac{\mu}{\lambda+\mu} & \frac{\lambda}{\lambda+\mu} \\ \frac{\mu}{\lambda+\mu} & \frac{\lambda}{\lambda+\mu} \end{bmatrix} \; ; \quad M_2 = \begin{bmatrix} \frac{\lambda}{\lambda+\mu} & \frac{-\lambda}{\lambda+\mu} \\ \frac{-\mu}{\lambda+\mu} & \frac{\mu}{\lambda+\mu} \end{bmatrix}$$

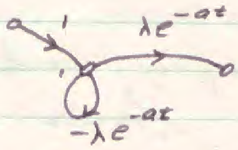
$$e^{\int_0^t A(t)dt} = M_1 + M_2 e^{-(\lambda+\mu)\left[\frac{1-e^{-at}}{a}\right]}$$

Note: As $a \to 0$, we get the same solution as before

$a \neq 0$, $\lim_{t\to\infty} e^{\int_0^t A(t)dt} = e^{-\frac{\lambda+\mu}{a}}$

$\mu = 0$, $t \to \infty$, steady state π_{12} $\Leftrightarrow$ no service.

$\pi_{12} = 1 - e^{-\frac{\lambda}{a}}$ $\Leftrightarrow$



This is the same problem as searching with the searcher getting tired.

Example: $A(t) = \begin{bmatrix} -\lambda e^{-at} & \lambda e^{-at} \\ \mu e^{-bt} & -\mu e^{-bt} \end{bmatrix}$

Repair rate slows at a different rate than the breakdown rate.

$\lambda_1 = 0$, $\lambda_2 = -\lambda e^{-at} - \mu e^{-bt}$

$$M_1 = \begin{bmatrix} \frac{\mu e^{-bt}}{\lambda e^{-at} + \mu e^{-bt}} & \frac{\lambda e^{-at}}{(\quad)} \\ \frac{\mu e^{-bt}}{(\quad)} & \frac{\lambda e^{-at}}{(\quad)} \end{bmatrix} ; \quad M_2 = \begin{bmatrix} \frac{\lambda e^{-at}}{(\,)} & -\frac{\lambda e^{-at}}{(\,)} \\ \frac{-\mu e^{-bt}}{(\,)} & \frac{\mu e^{-bt}}{(\,)} \end{bmatrix}$$

$$e^{\int_0^t A(t)dt} = M_1 + M_2 e^{\int_0^t(-\lambda e^{-at} - \mu e^{-bt})dt}$$

$$= M_1 + M_2 e^{-\frac{\lambda}{a}(1-e^{-at})} e^{-\frac{\mu}{b}(1-e^{-bt})}$$

$$\lim_{t\to\infty} e^{\int_0^t A(t)dt} = \begin{cases} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + e^{-\frac{\lambda}{a}} e^{-\frac{\mu}{b}} \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}, & \underline{\underline{a > b}} \\ \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} + e^{-\frac{\lambda}{a}} e^{-\frac{\mu}{b}} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, & \underline{\underline{a < b}} \end{cases}$$

Laplace Transforms in Continuous Markov Processes:

$$\dot{\underline{\pi}}(t) = \underline{\pi}(t)\, A(t)$$

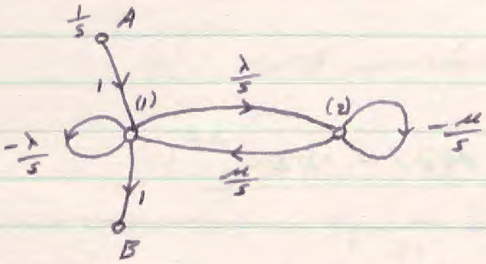
Let $A(t) = A$ be constant; $\quad \underline{\Pi}(s) \leftrightarrow \underline{\pi}(t)$

$$s\,\underline{\Pi}(s) - \underline{\pi}(0) = \underline{\Pi}(s)\, A$$

$$\boxed{\underline{\Pi}(s) = \underline{\Pi}(s)\frac{A}{s} + \frac{\underline{\pi}(0)}{s}}$$

Using the flow graph representation of the system, we must start the system off with a step probabilistically weighted according to $\underline{\pi}(0)$.

Example: One machine; State (1): machine working $\quad \lambda$ = breakdown rate
State (2): " broken $\quad \mu$ = repair rate



$$A = \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix}$$

$$T_{AB} = \frac{1 + \frac{\mu}{s}}{1 + \frac{(\lambda+\mu)}{s} - \frac{\lambda\mu}{s^2} + \frac{\lambda\mu}{s^2}} = \frac{s+\mu}{s+\lambda+\mu} \leftrightarrow$$

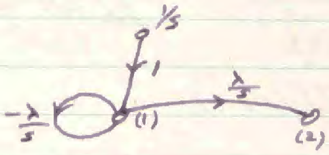
$$\Pi_1(s) = \frac{1}{s}T_{AB} = \frac{\frac{\mu}{\lambda+\mu}}{s} + \frac{\frac{\lambda}{\lambda+\mu}}{s+\lambda+\mu}$$

$$\pi_1(t) = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu}e^{-(\lambda+\mu)t}$$

Example: (1): No arrival has occured
(2): At least one arrival has occured

Poisson arrivals, arrival rate λ

$$A = \begin{bmatrix} -\lambda & \lambda \\ 0 & 0 \end{bmatrix}$$



$$\Pi_2(s) = \frac{1}{s}\,\frac{\frac{\lambda}{s}}{1+\frac{\lambda}{s}} = \frac{1}{s}\,\frac{\lambda}{s+\lambda} = 1 - e^{-\lambda t}$$

Let $p(t)$ = density function of inter-arrival time

$$p(t) = \dot{\Pi}_2(t)$$

$$p(s) = s\,\Pi_2(s) = \frac{\lambda}{s+\lambda} \leftrightarrow \underline{p(t) = \lambda e^{-\lambda t}}$$

We can find $\underline{\Pi}(t)$ directly by matrix algebra:

$$s\,\underline{\Pi}(s) - \underline{\Pi}(0) = \underline{\Pi}(s)\,A$$

$$\underline{\Pi}(s)\,[sI - A] = \underline{\Pi}(0)$$

$$\boxed{\underline{\Pi}(s) = \underline{\Pi}(0)\,[sI-A]^{-1}} \longleftarrow$$

Using the previous example,

$$A = \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix}$$

$$sI - A = \begin{bmatrix} s+\lambda & -\lambda \\ -\mu & s+\mu \end{bmatrix}$$

$$\det(sI-A) = s(s+\lambda+\mu)$$

$$[sI-A]^{-1} = \frac{1}{s(s+\lambda+\mu)}\begin{bmatrix} s+\mu & \lambda \\ \mu & s+\lambda \end{bmatrix}$$

$$[sI-A]^{-1} = \frac{1}{s}\underbrace{\begin{bmatrix} \frac{\mu}{\lambda+\mu} & \frac{\lambda}{\lambda+\mu} \\ \frac{\mu}{\lambda+\mu} & \frac{\lambda}{\lambda+\mu} \end{bmatrix}}_{S} + \frac{1}{s+\lambda+\mu}\underbrace{\begin{bmatrix} \frac{\lambda}{\lambda+\mu} & \frac{-\lambda}{\lambda+\mu} \\ \frac{-\mu}{\lambda+\mu} & \frac{\mu}{\lambda+\mu} \end{bmatrix}}_{T}$$

$$\underline{\pi}(t) = \underline{\pi}(0)\left[S + T e^{-(\lambda+\mu)t}\right]$$

<u>Example</u>: Two machines $\begin{cases} \lambda = \text{breakdown rate} \\ \mu = \text{repair rate} \end{cases}$

(1): both working
(2): one "
(3): none "

$\boxed{\text{Want } \pi_{13}(t)}$

$$A = \begin{bmatrix} -2\lambda & 2\lambda & 0 \\ \mu & -(\lambda+\mu) & \lambda \\ 0 & 2\mu & -2\mu \end{bmatrix}$$

$$(sI-A) = [\quad]$$

$$\det(sI-A) = s(s+\lambda+\mu)(s+2\lambda+2\mu)$$

$$(sI-A)^{-1}_{13} = \frac{2\lambda^2}{s(s+\lambda+\mu)(s+2\lambda+2\mu)} =$$

$$= \frac{\frac{\lambda^2}{(\lambda+\mu)^2}}{s} + \frac{\frac{-2\lambda^2}{(\lambda+\mu)^2}}{s+(\lambda+\mu)} + \frac{\frac{\lambda^2}{(\lambda+\mu)^2}}{s+2(\lambda+\mu)}$$

$$\pi_{13}(t) = \frac{\lambda^2}{(\lambda+\mu)^2}\left[1 - 2e^{-(\lambda+\mu)t} + e^{-2(\lambda+\mu)t}\right]$$

Dynamic Programming:

Assume that on a trip we know we have N service stations available & we know the prob. distribution of their gas prices:

$u \equiv$ utility or desirability (this is set by us, as, e.g., $u \sim \frac{1}{\text{price}}$)

$d_i \equiv$ critical desirability of next station given that i stations remain; i.e., that level below which we will not buy.

$v_i \equiv$ expected desirability of station patronized if i stations remain. This is a function of the decision policy we follow.

We want to find a policy for determining our d_i so that v_i is a max.

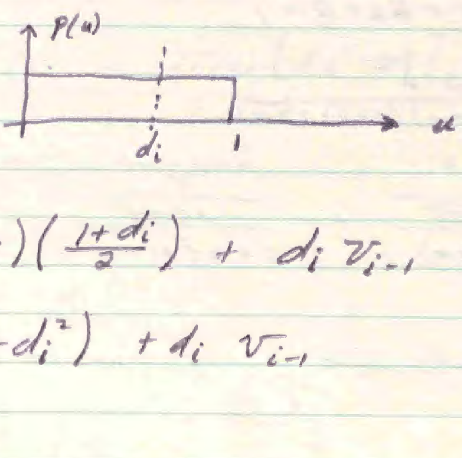
Suppose we fix d_i:

$$v_i = \underbrace{[\Pr\{\text{patronize next station}\}]}_{f(d_i)} \times \underbrace{[\text{expected desirability} \mid \text{we do patronize}]}_{f(d_i)}$$

$$+ \underbrace{[\Pr\{\text{don't patronize}\}]}_{f(d_i)} \times v_{i-1}$$

The "best" policy is the one which maximizes $v_i(d_i)$.

Example:



$$v_i = (1-d_i)\left(\frac{1+d_i}{2}\right) + d_i v_{i-1}$$

$$= \tfrac{1}{2}(1-d_i^2) + d_i v_{i-1}$$

Now we maximize v_i :

~~Note that this ... ...~~

$$\frac{\partial}{\partial d_i} v_i = -d_i + v_{i-1} = 0$$

$$\rightarrow \boxed{d_i = v_{i-1}}$$

If $d_i < v_{i-1}$, then we expect to make more on future stations & obviously won't buy.

$$\max_{d_i} v_i = \frac{1}{2}(1 - v_{i-1}^2) + v_{i-1}^2 = \underline{\frac{1}{2}(1 + v_{i-1}^2)}$$

To solve completely, we must know v_0. Suppose that $v_0 = 0$, i.e., we must buy gas on this trip.

i	$d_i = v_{i-1}$	$v_i = \frac{1}{2}(1+v_{i-1}^2)$
0	—	0
1	0	0.5
2	0.5	0.625
3	0.625	0.695
4	0.695	0.742
5	0.742	0.772
6	0.772	0.799
∞	1.0	1.0

Suppose $v_0 = 1$, we must buy at the last station. Obviously we will buy there:

i	d_i	v_i
0	—	1
1	1	1
2	1	1
3	1	1

Non-optimal policy; $v_0 = 0$: $d_i \equiv 0.5$

i.e., we buy if the desirability is above average.

$$v_i = \frac{1}{2}(1 - d_i^2) + d_i v_{i-1} = 0.375 + \frac{1}{2} v_{i-1}$$

i	d_i	v_i
0	—	0
1	0.5	0.375
2	0.5	0.563
3	0.5	0.656
4	0.5	0.703
5	0.5	0.727
∞	0.5	0.750

Note that with this policy, there is the possibility that we may not buy at all.
With the optimal policy, we will always buy.

Another policy: $d_i = \begin{cases} \frac{1}{2}, & i = 1, 2 \\ v_{i-1}, & i = 3, 4, \dots \end{cases}$ $v_0 = 0$

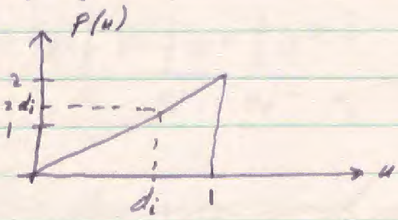
i	d_i	v_i
0	-	0
1	0.5	0.375
2	0.5	0.563
3	0.563	0.658
4	0.658	0.717
5	0.717	0.757
6	0.757	0.786
∞	1	1

$$v_i = \frac{1}{2}(1 + v_{i-1}^2)$$

This shows that a non-optimal policy can approach the optimal ~~policy~~ expected return if we use the optimal policy for large i.

Same problem, different desirability distribution.

Cheap gas more prevalent than expensive gas:



$$P(u) = 2u \quad , \; 0 \le u \le 1.$$

$$v_i = \frac{\int_{d_i}^{1} u \cdot 2u \, du}{1-d_i^2} (1-d_i^2) + d_i^2 v_{i-1}$$

$$= \tfrac{2}{3} - \tfrac{2}{3} d_i^3 + d_i^2 v_{i-1}$$

$$\underset{d_i}{\mathrm{Max}}\; v_i \rightarrow \frac{\partial v_i}{\partial d_i} = 0 \rightarrow d_i = v_{i-1}$$

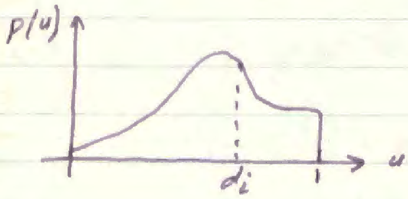
$$\underset{d_i}{\mathrm{Max}}\; v_i = \tfrac{1}{2}\left(2 + v_{i-1}^3\right)$$

Here the policy is the same as for the uniform distribution, but the rewards are different.

i	d_i	v_i (opt)
0	—	0
1	0	0.667
2	0.667	0.776
3	0.766	0.817
4	0.817	0.850
∞	1.0	1.0

$\Big\}\; v_0 = 0$

Optimal Policy for Arbitrary Distribution of Desirability:



$$v_i = \int_{d_i}^{1} u\,P(u)\,du + v_{i-1}\int_0^{d_i} P(u)\,du$$

For the optimal policy,

$$\frac{\partial v_i}{\partial d_i} = 0 = -d_i P(d_i) + v_{i-1} P(d_i)$$

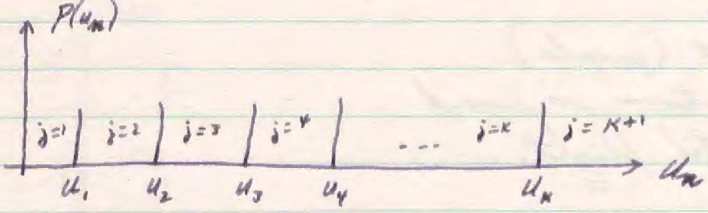
$\therefore$ $\boxed{d_i = v_{i-1}}$ is always the optimal policy.

This is just common sense. If we expect to find something more desirable later on, we don't buy ($v_{i-1} > d_i$). But if we don't expect to do better, we will buy ($v_{i-1} < d_i$).

The expected ~~#~~ values received are

$$\boxed{v_i = \int_{v_{i-1}}^{1} u\,P(u)\,du + v_{i-1}\int_0^{v_{i-1}} P(u)\,du}$$

Discrete Probability Distributions :



j^{th} region between u_{j-1} & u_j

Our decision level should be placed in region j :

$$v_i = \sum_{n=j}^{K} u_n P(u_n) + v_{i-1} \sum_{n=1}^{j-1} P(u_n)$$

To find the optimal policy, we must maximize the discrete function $v_i(j)$.

Let $f(n)$ be a discrete function with a single maximum :

$$\Delta f(n) \equiv f(n+1) - f(n)$$

$f(n)$ is a maximum for that value of n which is the smallest such that $\Delta f(n) < 0$.

To maximize the above v_i , we find

$$\Delta_j \left\{ \sum_{n=j}^{K} u_n P(u_n) + v_{i-1} \sum_{n=1}^{j-1} P(u_n) \right\}$$

$$= \sum_{n=j+1}^{K} u_n P(u_n) - \sum_{n=j}^{K} u_n P(u_n) + v_{i-1} \left[\sum_{n=1}^{j} P(u_n) - \sum_{n=1}^{j-1} P(u_n) \right]$$

$$= -u_j P(u_j) + v_{i-1} P(u_j)$$

For this quantity to be less than zero, we require

$$\boxed{u_j > v_{i-1}}$$

or $\boxed{d_i = v_{i-1}}$

Airline Reservations :

m = capacity of flight (people)
k = number of reservations on hand
n = " " days til flight.

Two policies : A: accept reservations
R: reject "

The policy is set for the whole day in the morning; cancellations are tallied & counted in at night.

→ $P(i) \equiv Pr\{i \text{ arrivals in a day}\}$

→ $q \equiv Pr\{\text{reserved customer will cancel on any given } \cancel{\text{night}} \text{ day}\}$

→ $P(j,k) \equiv Pr\{j \text{ cancellations from } k \text{ reservations in any given day}\}$

→ $p = 1 - q$

$$P(j,k) = \binom{k}{j} q^j p^{k-j}$$

→ $v(k,n) \equiv$ expected monetary return from k reservation at n days before the flight.

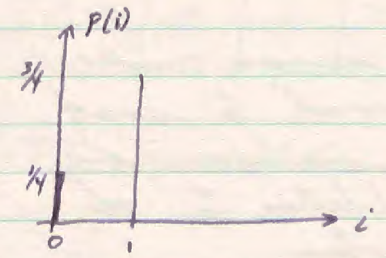
$v(m,0)$ = maximum value of v ; a boundary condition.

For the optimal policy :

$$\boxed{v(k,n) = \max_{A,R} \begin{cases} A: \sum_i P(i) \sum_{j=0}^{k+i} v(k+i-j, n-1)\, P(j, k+i) \\ R: \sum_{j=0}^{k} v(k-j, n-1)\, P(j,k) \end{cases}}$$

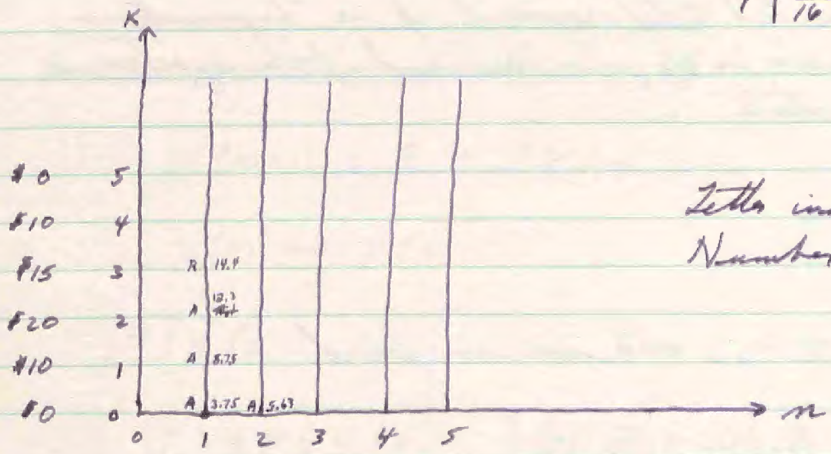
$\sum v(k+i, n-1) P(i) = Pr\{i \text{ arrivals}\}$

Example:



$q = p = \frac{1}{2}$

$k \downarrow$ $j \rightarrow$	0	1	2	3	4
0	1				
1	$\frac{1}{2}$	$\frac{1}{2}$			
2	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$		
3	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	
4	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{16}$



Letter indicates best policy for any point.
Number is expected return.

$$v(0,1) = \underset{A,R}{\text{Max}} \begin{Bmatrix} A: & \frac{1}{4}(0) + \frac{3}{4}\left[\frac{1}{2}(10) + \frac{1}{2}(10)\right] = 3.75 \\ R: & 0 \end{Bmatrix} = 3.75$$

$$v(1,1) = \underset{A,R}{\text{Max}} \begin{Bmatrix} A: & \frac{1}{4}(5) + \frac{3}{4}\left[\frac{1}{4}(20) + \frac{1}{2}(10) + \frac{1}{4}(0)\right] = 8.75 \\ R: & \frac{1}{2}(10) + \frac{1}{2}(0) = 5 \end{Bmatrix} = 8.75$$

$$v(2,1) = \underset{A,R}{\text{Max}} \begin{Bmatrix} A: & \frac{1}{4}(10) + \frac{3}{4}\left[\frac{1}{8}(15) + \frac{3}{8}(20) + \frac{3}{8}(10) + \frac{1}{8}(0)\right] = 12.3 \\ R: & \frac{1}{4}(20) + \frac{1}{2}(10) + \frac{1}{4}(0) = 10 \end{Bmatrix} = 12.3$$

Markovian Processes:

Suppose certain transitions are for some reason more valuable than others. We can assign a "reward" to each transition in accordance with their relative values. This defines a reward matrix $R = (r_{ij})$ in addition to the transition probability matrix for the system.

$v_i(n) \equiv$ expected <u>total</u> reward if n transitions remain in the process and if the system is in state i.

If we know r_{ij}, p_{ij}, and $v_i(0)$, then we can find

$$v_i(n+1) = \sum_{j=1}^{N} p_{ij}\left[r_{ij} + v_j(n)\right]$$

Let $\boxed{q_i \equiv \sum_{j=1}^{N} p_{ij} r_{ij}}$ = expected reward for one transition from ~~out of~~ state i.

so $$\boxed{v_i(n+1) = q_i + \sum_{j=1}^{N} p_{ij} v_j(n)} \qquad \underline{q} = PR$$

<u>Example</u>: Toymaker: (1): successful toy for the week
(2) unsuccessful " " " "

$$P = \begin{bmatrix} .5 & .5 \\ .4 & .6 \end{bmatrix} \quad ; \quad R = \begin{bmatrix} 9 & 3 \\ 3 & -7 \end{bmatrix}$$

$$\underline{q} = PR = \begin{bmatrix} 6 \\ -3 \end{bmatrix}$$

i	$v_i(0)$	$v_i(1)$	$v_i(2)$	$v_i(3)$	$v_i(4)$	$v_i(5)$
1	0	6	7.5	8.55	9.555	10.555
2	0	-3	-2.4	-1.44	-0.444	+0.5556

The apparent trend is $\lim_{n\to\infty} \{v_i(n) - v_i(n-1)\} = 1$

$$\lim_{n\to\infty} \{v_1(n) - v_2(n)\} = 10$$

Can we find these limits directly?

$$\underline{v}(n+1) = \underline{q} + P\underline{v}(n)$$

$$\underline{V}(z) \leftrightarrow \underline{v}(n)$$

$$z^{-1}[\underline{V}(z) - \underline{v}(0)] = \frac{1}{1-z}\underline{q} + P\underline{V}(z)$$

$$[I - zP]\underline{V}(z) = \frac{z}{1-z}\underline{q} + \underline{v}(0)$$

$$\boxed{\underline{V}(z) = \frac{z}{1-z}[I-zP]^{-1}\underline{q} + [I-zP]^{-1}\underline{v}(0)}$$

Now assume $\underline{v}(0) = 0$:

$$\boxed{\underline{V}(z) = \frac{z}{1-z}[I-zP]^{-1}\underline{q}}$$

Toymaker:

$$I - zP = \begin{bmatrix} 1-\frac{1}{2}z & -\frac{1}{2}z \\ -\frac{2}{5}z & 1-\frac{3}{5}z \end{bmatrix}$$

$$\det[I-zP] = (1-z)(1-\tfrac{1}{10}z)$$

$$[I-zP]^{-1} = \frac{1}{(1-z)(1-\frac{1}{10}z)}\begin{bmatrix} 1-\frac{3}{5}z & \frac{1}{2}z \\ \frac{2}{5}z & 1-\frac{1}{2}z \end{bmatrix}$$

$$= \frac{1}{1-z}\begin{bmatrix} \frac{4}{9} & \frac{5}{9} \\ 4/9 & 5/9 \end{bmatrix} + \frac{1}{1-\frac{1}{10}z}\begin{bmatrix} 5/9 & -5/9 \\ -4/9 & 4/9 \end{bmatrix}$$

$$= \frac{1}{1-z}S + T^*(z)$$

$$\frac{z}{1-z}\left[I-zP\right]^{-1} = \frac{z}{(1-z)^2}\begin{bmatrix} 4/9 & 5/9 \\ 4/9 & 5/9 \end{bmatrix} + \frac{z}{(1-z)(1-\frac{1}{10}z)}\begin{bmatrix} 5/9 & -5/9 \\ -4/9 & 4/9 \end{bmatrix}$$

$$\downarrow$$

$$\frac{10/9}{1-z} + \frac{-10/9}{1-\frac{1}{10}z}$$

$$F(n) = nS + \frac{10}{9}\left[1-\left(\tfrac{1}{10}\right)^n\right] T^*$$

$$v(n) = F(n)\,\underline{q} \qquad ; \quad \underline{q} = \begin{bmatrix} 6 \\ -3 \end{bmatrix}$$

$$v(n) = n\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{10}{9}\left[1-\left(\tfrac{1}{10}\right)^n\right]\begin{bmatrix} 5 \\ -4 \end{bmatrix}$$

$$\boxed{\begin{aligned} v_1(n) &= n + \frac{50}{9}\left[1-\left(\tfrac{1}{10}\right)^n\right] \\ v_2(n) &= n - \frac{40}{9}\left[1-\left(\tfrac{1}{10}\right)^n\right] \end{aligned}}$$

For large n, $\left.\begin{aligned} v_1(n) &\to n + \tfrac{50}{9} \\ v_2(n) &\to n - \tfrac{40}{9} \end{aligned}\right\} \begin{aligned} & v_i(n) - v_i(n-1) \to 1 \\ & v_1(n) - v_2(n) \to 10 \end{aligned}$

Now we go back & pick up the initial conditions:

$$V(z) = \frac{z}{1-z}\left[I-zP\right]^{-1}\underline{q} + \left[I-zP\right]^{-1}\underline{v}(0)$$

&

$$\left[I-zP\right]^{-1} = \frac{1}{1-z}S + T^*(z)$$

$$V(z) = \frac{z}{(1-z)^2}S\,\underline{q} + \frac{z}{1-z}T^*(z)\,\underline{q} + \frac{1}{1-z}S\,\underline{v}(0) + T^*(z)\,\underline{v}(0)$$

For large n, this goes to

$$v(n) = nS\,\underline{q} + T^*(1)\,\underline{q} + S\,\underline{v}(0)$$

Let $\underline{g} \equiv S \underline{q}$

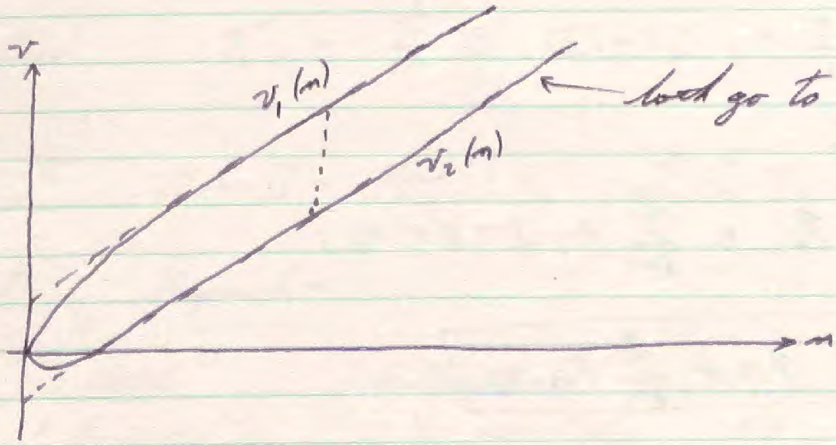
$\underline{v}(n) \equiv n \underline{g} + \underline{v}$

then $v_i(n) \to n g_i + v_i$ as $n \to \infty$

$g_i = \sum_{j=1}^{N} s_{ij} q_j \quad ; \quad s_{ij} = \pi_j$ for irreducible, single-chain processes.

$g_i = \sum_{j=1}^{N} \pi_j q_j = g$ independent of i

$g \equiv$ gain of the process = rate of increase of values



both go to slope g for large n.

In the previous example,

$$[I - zP]^{-1} = \frac{1}{1-z}\begin{bmatrix} 4/9 & 5/9 \\ 4/9 & 5/9 \end{bmatrix} + \frac{1}{1-\frac{1}{10}z}\begin{bmatrix} 5/9 & -5/9 \\ -4/9 & 4/9 \end{bmatrix}$$

$$= \frac{1}{1-z} S + T^*(z)$$

$$T^*(1) = \begin{bmatrix} 50/81 & -50/81 \\ -40/81 & 40/81 \end{bmatrix}$$

$$\underline{g} = S\,\underline{g} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\underline{v} = T^*(1)\,\underline{q} + \underbrace{S\,\underline{v}(0)}_{=0} = \begin{bmatrix} 50/9 \\ -40/9 \end{bmatrix}$$

$$\boxed{\underline{v}(n) = n \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 50/9 \\ -40/9 \end{bmatrix}}$$

For a single chain process, we have

$$v_i(n+1) = q_i + \sum_j p_{ij}\, v_j(n)$$

$$v_i(n) = n\,g + v_i$$

or $(n+1)\,g + v_i = q_i + \sum_{j=1}^{N} p_{ij} \left[n\,g + v_j \right]$

$$\boxed{g + v_i = q_i + \sum_{j=1}^{N} p_{ij}\, v_j}$$

If we add a constant to v_i, the same equation results. So ∴ one of the v_i is arbitrary. In particular, let $v_N = 0$.

$\boxed{\text{Example}}$: $P = \begin{bmatrix} 0.5 & 0.5 \\ 0.4 & 0.6 \end{bmatrix}$; $\underline{q} = \begin{bmatrix} 6 \\ -3 \end{bmatrix}$

$v_2 = 0$

$$\left.\begin{aligned} g + v_1 &= 6 + 0.5\,v_1 \\ g &= -3 + 0.4\,v_1 \end{aligned}\right\} \qquad \boxed{\begin{aligned} g &= 1 \\ v_1 &= 10 \\ v_2 &= 0 \end{aligned}}$$

Decision-Making:

Assume K alternatives are open to us. Then we have p_{ij}^k and r_{ij}^k which are functions of the alternative chosen.

Let $d_i(n)$ be a decision & $\underline{d}(n)$ the associated policy

$v_i(n)$ = total expected return in n moves from state i

For the optimal policy,

$$v_i(n+1) = \underset{k}{Max} \sum_{j=1}^{N} p_{ij}^k \left[r_{ij}^k + v_j(n) \right]$$

Define $$q_i^k \equiv \sum_{j=1}^{N} p_{ij}^k r_{ij}^k$$

&

$$v_i(n+1) = \underset{k}{Max} \left\{ q_i^k + \sum_{j=1}^{N} p_{ij}^k v_j(n) \right\}$$

Example: Toymaker

State	Alternatives	p_{i1}^k	p_{i2}^k	r_{i1}^k	r_{i2}^k	q_i^k
1. Successful toy	1. No advertising	0.5	0.5	9	3	6 ←
	2. Advertising	0.8	0.2	4	4	4
2. Unsuccessful toy	1. No research	0.4	0.6	3	−7	−3 ←
	2. Research	0.7	0.3	1	−19	−5

We want to know what policy will maximize his return. We start with the policy which will maximize his immediate return: i.e., choose alternative (1) in either state:

State(i)	$d_i(1)$	$v_i(1)$	$d_i(2)$	$v_i(2)$	$d_i(3)$	$v_i(3)$	$d_i(4)$	$v_i(4)$
1	1	6	2	8.2	2	10.22	2	12.222
2	1	−3	2	−1.7	2	0.23	2	2.223

In this example,

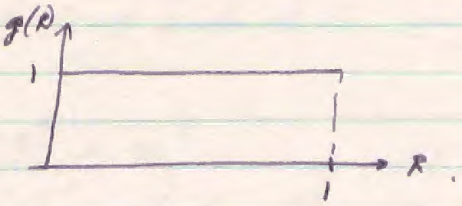
$$\left\{\begin{array}{l} v_i(n) - v_i(n-1) = 2 \\ v_1(n) - v_2(n) = 10 \end{array}\right\}$$

$$g = \sum_{i=1}^{N} \pi_i q_i$$

$$\max_k \left\{ q_i^k + \sum_j p_{ij}^k (v_j + ng) \right\}.$$

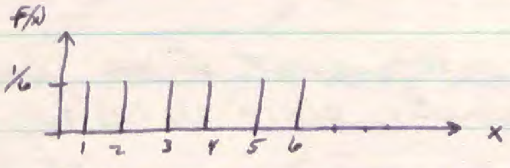
Simulation :

Suppose we can select a number $0 \le R \le 1$ according to the probability density function:

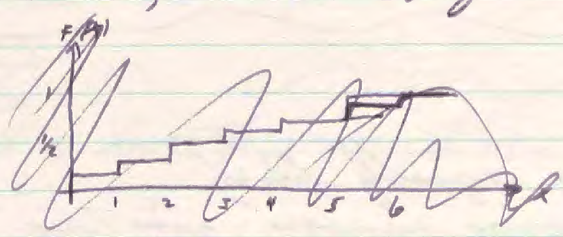


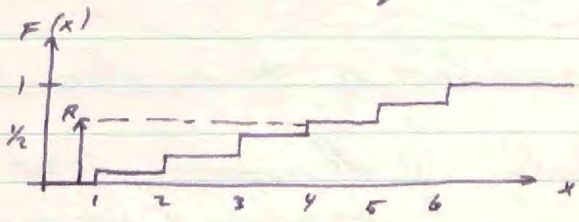
From this we want to generate ~~an~~ selection from other distributions so as to simulate a physical event.

Example: Suppose we want to simulate the rolling of a die, where the distribution is:



To do this, we can compute $F(x) = \int_0^x f(x)\,dx$ and get



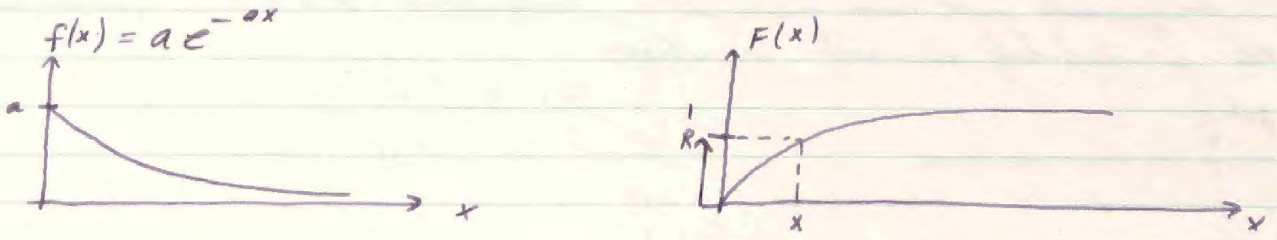


Now we lay out R along $F(x)$; $R \le 1$. If $\frac{1}{6} < R < \frac{2}{6}$, the number rolled on the die is 2.

In general:

$$\left.\begin{aligned} P_r\{x_1 < x < x_2\} &= \int_{x_1}^{x_2} f(x)\,dx = F(x_2) - F(x_1) \\ &= P_r\{F(x_1) < R < F(x_2)\} \end{aligned}\right\} \text{ for any } f(x).$$

Example: Suppose we want to select samples according to an exponential distribution:



$f(x) = a e^{-ax}$
$F(x) = 1 - e^{-ax} = R$

We are choosing $R = F(x)$ at random, so we have

$x = \frac{1}{a} \ln \frac{1}{1-R}$ gives a sample from the distribution $f(x) = a e^{-ax}$ if R is chosen from a uniform distribution $0 < R < 1$.

But R is symmetrically distributed with respect to the function $1-R$, so we can equally well write $g(R) = g(1-R) = 1$ &

$x = \frac{1}{a} \ln \frac{1}{R}$

Generation of the samples R:

Ideally, (1) all numbers $0 \le R \le 1$ are equally likely
(2) knowledge of any or all previous numbers gives no information on the next number (i.e., samples are independent)

Any real physical process is, however, deterministic, so we cannot really generate this ideal sampling. We can, however, get very close:

e.g.: Congruential series:

$R_{n+1} = M R_n \pmod{B}$.

If M and B are large enough, we can get a very good <u>pseudo-random</u> selection of numbers.

Convenient choices of M + B for use on the Marchant 20 digit ~~hand~~ desk calculator are

$$M = 7^{11} \quad \text{(has 10 digits)}$$
$$B = 10^{10}$$

Select R_0 as any odd number not ending in 5. Multiply $R_0 \times 7^{11}$ which yields a 20 digit number. Take the last 10 digits as the first random number.

<u>Characteristics of the series:</u>

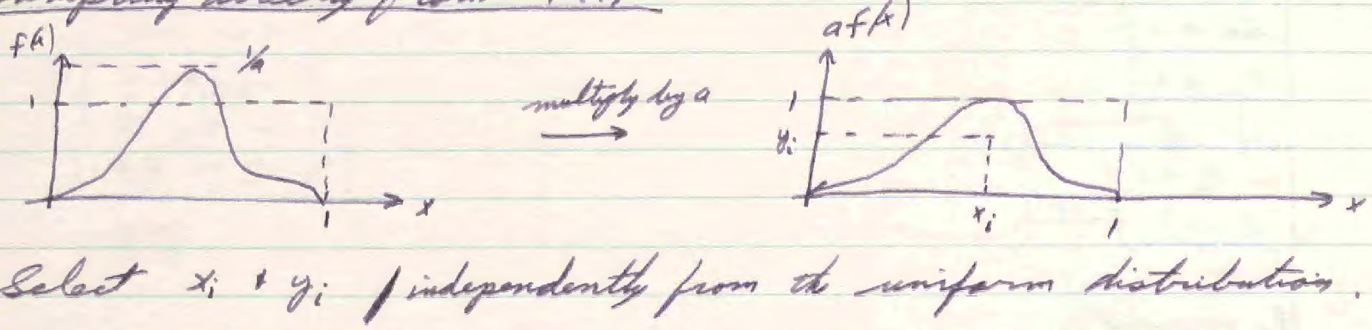
(1) Series contains a large # of numbers (here, 50×10^6) before it repeats itself.

(2) # of R numbers in Δx is proportional to Δx if Δx is not too small.

(3) Knowledge of the initial digits of a random number does not limit the succeeding numbers.

(4) If we want two sets of random samples, we must pick two starting points.

<u>Sampling directly from $f(x)$</u>:



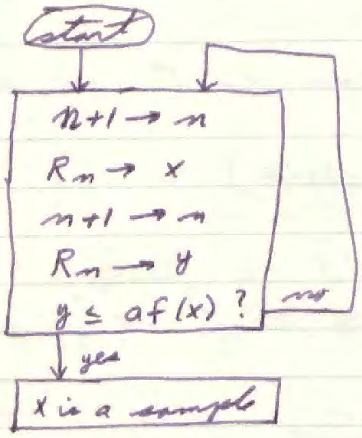
Select x_i + y_i independently from the uniform distribution.

Accept x_i as a point sample from $f(x)$ if $y_i \le a\,f(x_i)$
Reject x_i - - - - - - - - $y_i > a\,f(x_i)$.

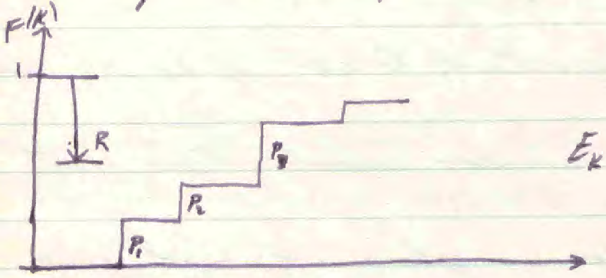
On the average aN acceptable samples will result from N trials (x_i, y_i).

So it will take $\left(\frac{2}{a}N\right)$ R-selections to get N samples.

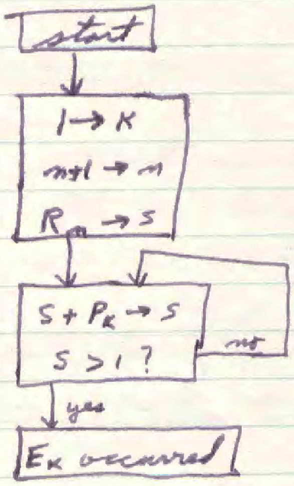
<u>Computer programs to generate samples</u>:



Consider a <u>finite</u> set of events E_K, probabilities P_K.



$$E_K \text{ occurs if } \begin{cases} R + P_1 + P_2 + \dots + P_{K-1} < 1 \\ \& \; R + P_1 + P_2 + \dots P_K > 1. \end{cases}$$

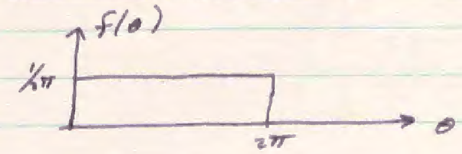


Samples from Poisson:

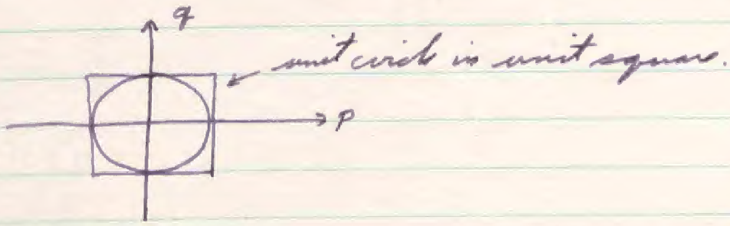
$$P(n) = e^{-\mu}\frac{\mu^n}{n!} \quad \& \quad P(n+1) = \frac{\mu}{n+1} P_n$$

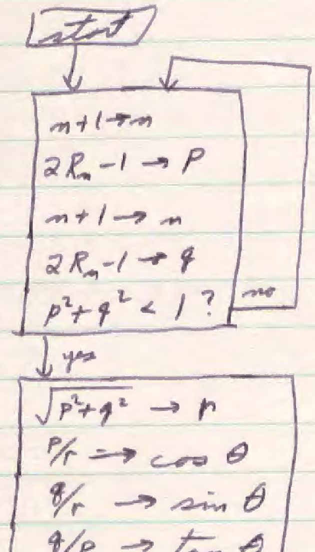
In a computer, let $P_0 = 1$ & work out $P(n)$ recursively.

Example:



What is the distribution of $\sin\theta$? of $\cos\theta$? of $\tan\theta$?





Wiener
Bloomfield
Brennan

1. New & unfamiliar subject
2. Want to talk to people before writing def prop.

Approach: Suppose I had to draw up an insp system; what would I consider & how would I start?

1. What will I look for?
2. How much will it cost?
3. What equip-people-etc. mix will I use?
4. My goal is to detect hidden stockpiles & factories & to detect unallowed use of open factories.

λ Poisson $\longrightarrow \boxed{\tfrac{1}{\mu}} \longrightarrow$

$P(n,t) = \Pr\{n \text{ in syst at time } t\}$.

$$P(n, t+dt) = \mu\, dt\, P(n+1,t) + [1-\lambda dt][1-\mu dt]P(n,t) + \lambda dt\, P(n-1,t)$$

$$= \mu\, dt\, P(n+1,t) + [1-(\lambda+\mu)dt]P(n,t) + \lambda dt\, P(n-1,t)$$

$$\frac{dP(n,t)}{dt} = \mu P(n+1,t) - (\lambda+\mu)P(n,t) + \lambda P(n-1,t)$$

$$P(0,t+dt) = \mu\, dt\, P(1,t) + (1-\lambda dt)P(0,t)$$

$$\frac{d}{dt}P(0,t) = \mu P(1,t) - \lambda P(0,t)$$

steady state: $P(n) = \left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n = (1-\rho)\rho^n$

def $P(z,s) = \sum_{n=0}^{\infty} z^n \int_0^{\infty} P(n,t)e^{st}\,dt$

If $\mu = 0$, $P(z,s) = \dfrac{\lambda}{s+\lambda-\lambda z}$

$G(n,s) = \int_0^{\infty} P(n,t)e^{-st}\,dt$, $H(z,t) = \sum z^n P(n,t)$

$P(z,s) = \sum z^n G(n,s) = \int H(z,t)e^{-st}\,dt$

$$P(z,s) = \frac{\lambda}{s+\lambda-\lambda z} \longleftrightarrow \frac{(\lambda t)^n e^{-\lambda t}}{n!}$$

Try to apply 2 dim xfm for orig diff eqs

$$\frac{d\,P(n+1,t)}{dt} = \mu\, P(n+2,t) - (\lambda+\mu)\, P(n+1,t) + \lambda\, P(n,t)$$

$$\mathcal{Z}[P(n+1,t)] = z^{-1}[\mathcal{Z}\, P(n,t) - P(0,t)]$$

$$\mathcal{T} = \mathcal{L}\mathcal{Z}[P(n+1,t)] = z^{-1}[P(z,s) - G(0,s)]$$

$$\mathcal{Z}[P(n+2,t)] = z^{-2}[\mathcal{Z}P(n,t) - P(0,t) - zP(1,t)]$$

$$\mathcal{T}P(n+2,t) = \mathcal{L}[\mathcal{Z}\, P(n+2,t)] = z^{-2}[P(z,s) - G(0,s) - z\,G(1,s)]$$

$$\frac{d}{dt}\mathcal{Z}[P(n+1,t)] = z^{-1}\mathcal{Z}\frac{d}{dt}P(n,t) - \frac{d}{dt}P(0,t)$$

$$\mathcal{L}[\qquad] = z^{-1}[\mathcal{Z}(sG(n,s) - P(n,0)) - s\,G(0,s) + P(0,0)]$$

$$= z^{-1}\{s\,P(z,s) - H(z,0) - s\,G(0,s) + P(0,0)\}$$

Now eq becomes in xfm domain:

$$s z^{-1} P(z,s) - z^{-1} H(z,0) - s z^{-1} G(0,s) + z^{-1} P(0,0) = \mu z^{-2} P(z,s) - \mu z^{-2} G(0,s) - \mu z^{-1} G(1,s)$$
$$- \lambda z^{-1} P(z,s) + \lambda z^{-1} G(0,s)$$
$$- \mu z^{-1} P(z,s) + \mu z^{-1} G(0,s)$$
$$+ \lambda P(z,s)$$

~~$-(\lambda+\mu)$~~

If Laplace eq for $n=0$, get

$$s\,G(0,s) - P(0,0) = \mu\, G(1,s) - \lambda\, G(0,s)$$

this occurs in eq for $n \geq 1$ so:

$$G(0,s) = \frac{1/s}{1 + \lambda/s - Q(s)} = \frac{1}{s+\lambda - sQ(s)}$$

$$= \frac{1}{\frac{1}{2}(s+\lambda-\mu) + \frac{1}{2}\sqrt{}}$$

$$= \frac{2}{s+\lambda-\mu+\sqrt{(s+\lambda+\mu)^2 - 4\lambda\mu}}$$

$$P(z,s) = \frac{z + \mu(z-1)\,\dfrac{2}{s+\lambda-\mu+\sqrt{(s+\lambda+\mu)^2-4\lambda\mu}}}{-\mu + (s+\lambda+\mu)z - \lambda z^2}$$

If $\mu = 0$, should get Poisson:

get $P(z,s) = \dfrac{z + \dfrac{2}{s+\lambda+\sqrt{(s+\lambda)^2}}}{(s+\lambda)z - \lambda z^2} = \dfrac{z + \dfrac{1}{s+\lambda}}{(s+\lambda)z - \lambda z^2}$

in gen $$P(n,t) = e^{-(\lambda+\mu)t}\Big[\Big(\sqrt{\tfrac{\mu}{\lambda}}\Big)^{-n} I_{-n}\big(2\sqrt{\lambda\mu}\,t\big) + \Big(\sqrt{\tfrac{\mu}{\lambda}}\Big)^{1-n} I_{1+n}\big(2\sqrt{\lambda\mu}\,t\big)$$
$$+ (1-\rho)\rho^n \sum_{k=n+2}^{\infty}\Big(\sqrt{\tfrac{\mu}{\lambda}}\Big)^k I_k\big(2\sqrt{\lambda\mu}\big)t\Big]$$

$$P(0,t) = e^{-\lambda t}\Big[I_0\big(2\sqrt{\lambda\mu}\,t\big) + \sqrt{\tfrac{\mu}{\lambda}}\, I_1\big(2\sqrt{\lambda\mu}\,t\big)$$
$$+ (1-\rho)\sum_{k=2}^{\infty}\Big(\sqrt{\tfrac{\mu}{\lambda}}\Big)^k I_k\big(2\sqrt{\lambda\mu}\,t\big)\Big]$$

$$P(z,s)\left[sz+\lambda z+\mu z-\mu-\lambda z^2\right] = z H(z,0) + G(0,s)\left[\cancel{sz} - \mu + \cancel{\lambda\mu} + \mu z\right] \cancel{- \mu z G(1,s)} - z P(0,0)$$

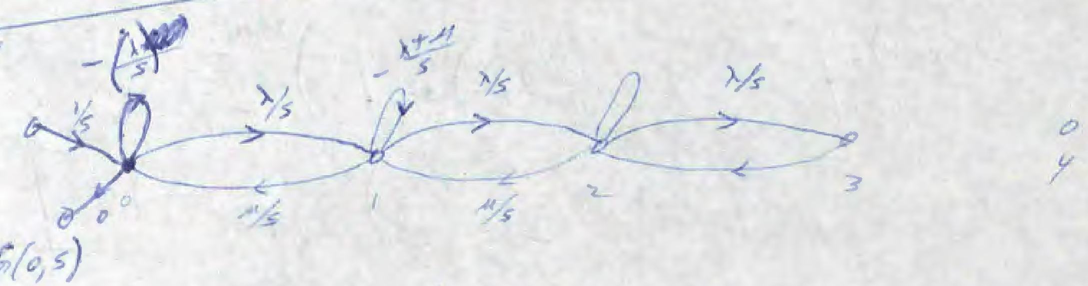
$$P(z,s) = \frac{z H(z,0) + \mu(z-1)G(0,s)}{-\mu + (s+\lambda+\mu)z - \lambda z^2}$$

Bound condy: $P(0,0)=1$, $P(n,0)=0,\ n\geq 1$ } syst started empty.

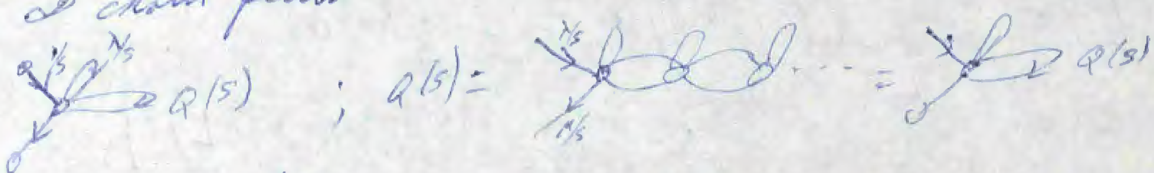
$$P(z,s) = \frac{z + \mu(z-1)G(0,s)}{-\mu + (s+\lambda+\mu)z - \lambda z^2}$$

$$\underset{z=1}{P(1,s)} = \int_0^\infty \left[\sum_{n=0}^{\infty} P(n,t)\right] e^{-st}\,dt = \frac{1}{s}$$

$$\underset{z=0}{P(0,s)} = G(0,s) \quad : \text{How do we find } G(0,s)\ ?$$



Use ∞ chain feedback $Q(s)$



$$\text{or } Q(s) = \frac{\lambda\mu/s^2}{1+\frac{\lambda+\mu}{s} - Q(s)} = \frac{\lambda\mu}{s^2+(\lambda+\mu)s - s^2Q(s)}$$

$$s^2Q(s) + (\lambda+\mu)sQ(s) - s^2Q^2(s) - \lambda\mu = 0 = s^2Q^2(s) - \left[s^2+s\lambda+s\mu\right]Q(s) + \lambda\mu$$

$$Q(s) = \frac{1}{2s^2}\left[s^2+s\lambda+s\mu \pm \sqrt{(s^2+s\lambda+s\mu)^2 - 4\lambda\mu s^2}\right]$$

$$= \frac{1}{2s}\left[s+\lambda+\mu \pm \sqrt{(s+\lambda+\mu)^2 - 4\lambda\mu}\right]$$